MATH236001

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH236001**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH2360
(January 2007)

Vector Calculus and Applications

Time allowed: **2 hours**

Answer FOUR of the FIVE questions.

All questions carry equal marks.

1. (a) The curve C is given parameterically by $x = t\cos t$, $y = t\sin t$ and $z = 2t$, for $0 \leq t \leq 2\pi$. Find the position vector of the end points of the curve, and then calculate the line integral
$$\int_C z\,ds.$$

(b) For the coordinate transformation,
$$u = x + y \qquad v = \frac{y}{x},$$
find x and y in terms of u and v and hence calculate the Jacobian of the transformation from (x, y) to (u, v) coordinates.
Sketch the region A in the (x, y) plane satisfying $x \leq y \leq 2x$ and $1 - x \leq y \leq 2 - x$, and the corresponding region A' in the (u, v) plane. By transforming to (u, v) coordinates evaluate the integral
$$\iint_A \frac{1}{xy}\,dx\,dy.$$

(c) Sketch the region of integration for the integral
$$I = \int_0^1 \int_{2y}^2 \exp\left(-\frac{1}{2}x^2\right)\,dx\,dy\ .$$
By interchanging the order of integration find the value of I.

2. (a) Consider the surface given by

$$f(x,y,z) = (x+z)\sin(x) + yz\exp(yz) - \pi = 0\ .$$

Find ∇f and calculate the unit normal vector to the surface at the point $(\frac{\pi}{2}, 0, 1)$. Find the equation for the tangent plane at this point and the shortest distance from this plane to the origin.

(b) Show using index notation that

$$\nabla \times (\nabla f) = 0.$$

Write $\nabla \times (g\nabla f)$ in index notation and show using index notation that

$$\nabla \times (g\nabla f) = -\nabla \times (f\nabla g).$$

for all scalar fields f and g and verify this result for

$$f = (x+y)^2 \text{ and } g = \frac{z}{x+y}.$$

Hence deduce that $\mathbf{F} = f\nabla g + g\nabla f$ is a conservative vector field for all scalar fields f and g. and find the corresponding potential field.

3. (a) Let $\mathbf{F}(x,y,z)$ be the vector field $\mathbf{F} = (x^3y, y+e^z, yz)$ and $f(x,y,z)$ be the scalar field $f = x^{3/2}\sin(y+z)$.
Calculate ∇f, $\nabla \cdot \mathbf{F}$, $\nabla \times \mathbf{F}$, $(\mathbf{F}\cdot\nabla)f$, $(\mathbf{F}\cdot\nabla)\mathbf{F}$.

(b) Explain what is meant by a conservative vector field. Show that the vector field

$$\mathbf{F}(x,y,z) = \left(\frac{y^2}{1+x},\ 2y\log(1+x),\ z^2\right)$$

obeys $\nabla \times \mathbf{F} = 0$. Find a corresponding potential field $\Phi(x,y,z)$ satisfying $\mathbf{F} = \nabla\Phi$. Evaluate the line integral

$$\int_P^Q \mathbf{F}\cdot d\mathbf{x}$$

over an arbitrary curve from $P = (0,1,1)$ to $Q = (1,1,2)$.

4. (a) Let S be the surface of the cone $z + \sqrt{x^2+y^2} = 2$ for $0 \le z \le 2$. Using cylindrical polar coordinates, or otherwise, calculate directly the surface integral

$$I = \iint\limits_S \mathbf{F}\cdot d\mathbf{S},$$

where $\mathbf{F} = (x, y, z^2+1)$.

QUESTION 4 CONTINUED...

(b) Calculate $\nabla \cdot \mathbf{F}$, for the function defined in part (a), and calculate

$$J = \iiint\limits_V (\nabla \cdot \mathbf{F})\, dV,$$

where V is the volume enclose by the surface S and the disc, $D_0 = \{(x, y, z) : x^2 + y^2 \leq 4, z = 0\}$.

(c) State the divergence theorem and show that it is verified in this case. Hint: remember to include the contribution from the disc D_0.

5. (a) Using the parameterisation $x = \cos\phi, y = \frac{1}{2}\sin\phi, z = 0$ or otherwise, calculate the closed line integral,

$$\oint\limits_C \mathbf{F} \cdot d\mathbf{x},$$

where $\mathbf{F} = (y + z - zy, -x + y + xz, x + y)$ and C is the ellipse $x^2 + 4y^2 = 1$, $z = 0$ transversed in an anticlockwise direction.

Hint: You may assume the following trigonometric identity:

$$2 \sin u \cos u = \sin 2u.$$

(b) Calculate $\nabla \times \mathbf{F}$ for the function given in part (a) and using the parameterisation

$$x = \sin\theta\cos\phi, \quad y = \frac{1}{2}\sin\theta\sin\phi, \quad z = \cos\theta, \text{ for } 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq 2\pi$$

evaluate

$$\iint\limits_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

where S is the spheroidal surface $x^2 + 4y^2 + z^2 = 1,\ 0 \leq z \leq 1$.

Hint: You may find easier to perform the ϕ integration first.

(c) State Stokes' theorem and verify it by comparing your answers to parts (a) and (b). Explain how Stokes' theorem allows you to find the value of the following integral without further calculations,

$$\iint\limits_{S'} \mathbf{G} \cdot d\mathbf{S},$$

where S' is the surface $x^2 + 4y^2 + z = 1$ for $0 \leq z \leq 1$ and $\mathbf{G} = (1 - x, -y, 2(z - 1))$.

END