MATH236001

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH236001**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH2360
(January 2006)

Vector Calculus and Applications

Time allowed: **2 hours**

Answer FOUR of the FIVE questions.

All questions carry equal marks.

1. (a) Calculate

$$\int_C \frac{x}{x+y+2z}\, ds$$

where s is the arc length parameter and C is the curve given by $x = t$, $y = 2-t$, $z = \frac{1}{2}t^2$, for $0 \le t \le 2$.

(b) For the coordinate transformation,

$$u = xy \qquad v = \frac{y}{x^2},$$

find x and y in terms of u and v and hence calculate the Jacobian of the transformation from (x, y) to (u, v) coordinates.
Sketch the region A in $x > 0$, $y > 0$ satisfying $x^2 \le y \le 2x^2$ and $1/x \le y \le 2/x$, and by transforming to (u, v) coordinates evaluate the integral

$$\iint_A \frac{y^2}{x^4} \exp(xy)\, dx\, dy.$$

(c) Sketch the region of integration for the integral

$$I = \int_1^e \int_{\ln(y)}^1 \frac{\exp(-x^2)}{y}\, dx\, dy\ .$$

By interchanging the order of integration find the value of I.

2. (a) Consider the surface given by

$$f(x,y,z) = x^2\sin(y) + \frac{z^2}{x} - 6 = 0 \ .$$

Find ∇f and calculate the unit normal vector to the surface at the point $(2, \frac{\pi}{2}, 2)$. Find the equation for the tangent plane and the shortest distance from this plane to the origin.

(b) Prove the following vector identity using index notation,

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0,$$
$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2\mathbf{u}.$$

By substituting $\mathbf{u} = \nabla \times \mathbf{F}$ show that

$$\nabla \times (\nabla \times (\nabla \times \mathbf{F})) = -\nabla^2(\nabla \times \mathbf{F}).$$

3. (a) Let $\mathbf{F}(x,y,z)$ be the vector field

$$\mathbf{F} = (\cos(x) + \sin(z),\ y\sin(x),\ x + e^z)$$

and $f(x,y,z)$ be the scalar field $f = z + \sqrt{x^2+y^2}$.
Calculate ∇f, $\nabla \cdot \mathbf{F}$, $\nabla \times \mathbf{F}$, $(\mathbf{F}\cdot\nabla)f$, $(\mathbf{F}\cdot\nabla)\mathbf{F}$.

(b) State the definition of a conservative vector field. Show that the vector field

$$\mathbf{F}(x,y,z) = \left(x\cos[\pi(x^2+y^2)] + x,\ y\cos[\pi(x^2+y^2)] - 2,\ \frac{1}{z}\right)$$

obeys $\nabla\times\mathbf{F} = 0$, and find obtain a potential field $\Phi(x,y,z)$ such that $\mathbf{F} = \nabla\Phi$. What can you conclude about the line integral

$$\int_P^Q \mathbf{F}\cdot d\mathbf{x}\,?$$

By using the potential field $\Phi(x,y,z)$, or otherwise, evaluate this integral from $P = (0,0,1)$ to $Q = (1,3,2)$.

4. (a) Sketch the surface, S given by $z + \sqrt{x^2+y^2} = 2$ for $0 \le z \le 1$. Using cylindrical polar coordinates, or otherwise, calculate directly the surface integral

$$I = \iint_S \mathbf{F}\cdot d\mathbf{S},$$

where $\mathbf{F} = (x, y, z + x^2 + y^2)$.

(b) Calculate $\nabla\cdot\mathbf{F}$, for the function defined in part (a), and calculate

$$J = \iiint_V (\nabla\cdot\mathbf{F})\,dV,$$

where V is the volume enclose by the surface S of part (a) together with the discs, $D_0 = \{(x,y,z) : x^2+y^2 \le 4, z = 0\}$ and $D_1 = \{(x,y,z) : x^2+y^2 \le 1, z = 1\}$.

(c) State the Divergence theorem and by comparing the values of I and J together with the surface integral over the discs, D_0 and D_1, show that it is verified in this case.

5. (a) Using the parameterisation $x = \frac{1}{2}\cos\phi, y = \frac{1}{2}\sin\phi, z = 0$ or otherwise, calculate the closed line integral,

$$\oint_C \mathbf{F} \cdot d\mathbf{x},$$

where $\mathbf{F} = (x^2 - 4y(x^2 + y^2), x - z, 0)$ and C is the circle $4x^2 + 4y^2 = 1,\ z = 0$ transversed in an anticlockwise direction.

(b) Calculate $\nabla \times \mathbf{F}$ and evaluate

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S},$$

where S is the surface of the parabaloid $4x^2 + 4y^2 + z = 1,\ 0 \le z \le 1$.

Hint: Use cylindrical polar coordinates to parameterise the surface. You may assume the following trigonometric identities:

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u), \qquad \cos^2 u = \frac{1}{2}(1 + \cos 2u).$$

(c) State Stokes' theorem and verify it by comparing your answers to parts (a) and (b). Explain how Stokes' theorem allows you to find the value of the following integral without further calculations,

$$\iint_{S'} \mathbf{G} \cdot d\mathbf{S},$$

where S' is the surface of the hemisphere $4x^2 + 4y^2 + 4z^2 = 1$ for $0 \le z \le \frac{1}{2}$ and $\mathbf{G} = (1, 0, 1 + 4x^2 + 12y^2)$.

END