MATH236001

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH236001**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH2360

(January 2005)

Vector Calculus and Applications

Time allowed: **2 hours**

Answer FOUR of the FIVE questions.

All questions carry equal marks.

1. (a) Calculate

$$\int_C \frac{x}{5y-4z}\,ds$$

where s is the arc length parameter and C is the curve given by $x = t$, $y = 1$, $z = 1 - t^2$, for $0 \le t \le 1$.

(b) Find the Jacobian of the transformation from (x, y) to (R, θ) coordinates given by

$$x = R\cos\theta, \qquad y = \frac{1}{2}R\sin\theta.$$

Hence evaluate the integral

$$\iint_A \frac{x^2}{x^2+4y^2}\,dx\,dy,$$

where A is the region between the ellipses $x^2 + 4y^2 = 1$ and $x^2 + 4y^2 = 4$.
Hint: You may assume the following trigonometric identity:

$$\cos^2 u = \frac{1}{2}\left(1 + \cos 2u\right).$$

(c) Sketch the region of integration for the integral

$$I = \int_0^2 \int_{\sqrt{2y}}^2 \exp(x^3)\,dx\,dy\ .$$

By interchanging the order of integration find the value of I.

2. (a) Consider the surface given by

$$f(x,y,z) = x^2 - \sin(\pi xy) + 2zy - 1 = 0 \ .$$

Find ∇f and calculate the unit normal vector to the surface at the point $(1, \frac{1}{2}, 1)$. Find the equation for the tangent plane and the shortest distance from this plane to the origin.

(b) Using index notation prove the following vector identity,

$$(\nabla \times \mathbf{u}) \times \mathbf{u} = (\mathbf{u} \cdot \nabla)\mathbf{u} - \tfrac{1}{2}\nabla\,(\mathbf{u} \cdot \mathbf{u})\,.$$

Verify this formula for the vector $\mathbf{u} = (x^2 + y^2, 2xy, x)$.

3. (a) Let $\mathbf{F}(x,y,z)$ be the vector field $\mathbf{F} = (x^2\sin(y), ye^z, zx)$ and $f(x,y,z)$ be the scalar field $f = (x+y-z)^2$.
Calculate ∇f, $\quad \nabla \cdot \mathbf{F}$, $\quad \nabla \times \mathbf{F}$, $\quad (\mathbf{F} \cdot \nabla)f$, $\quad (\mathbf{F} \cdot \nabla)\mathbf{F}$.

(b) Explain what is meant by a conservative vector field. Show that the vector field

$$\mathbf{F}(x,y,z) = \big(z + y^2 e^x,\ 2ye^x,\ x\big)$$

obeys $\nabla \times \mathbf{F} = 0$. What can you conclude about the line integral

$$\int_P^Q \mathbf{F} \cdot d\mathbf{x}\,?$$

By evaluating this integral from the origin to a general point, P, or otherwise, find a corresponding potential field $\Phi(x,y,z)$.

4. (a) Using cylindrical polar coordinates or otherwise calculate directly the surface integral

$$I = \iint\limits_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F} = (x, y, 2z-1)$ and S is the surface given by $z = 2 - x^2 - y^2$ for $0 \le z \le 2$.

(b) Calculate $\nabla \cdot \mathbf{F}$, for the function defined in part (a), and calculate

$$J = \iiint\limits_V (\nabla \cdot \mathbf{F})\, dV,$$

where V is the volume enclose by the surface S of part (a) and the disk, $x^2 + y^2 \le 2,\ z = 0$.

(c) State the Divergence theorem and by comparing the values of I and J show that it is verified in this case. **Hint**: ensure that you include the contribution from the disk, $x^2 + y^2 \le 2,\ z = 0$.

5. (a) Using the parameterisation $x = \cos\phi, y = \sin\phi, z = 0$ or otherwise, calculate the closed line integral,

$$\oint_C \mathbf{F} \cdot d\mathbf{x},$$

where $\mathbf{F} = (-y + x^2 y, -xy^2, 0)$ and C is the circle $x^2 + y^2 = 1,\ z = 0$ transversed in an anticlockwise direction.

Hint: You may assume the following trigonometric identities:

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u), \qquad \sin u \cos u = \frac{1}{2}\sin 2u.$$

(b) Calculate $\nabla \times \mathbf{F}$ and evaluate

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S},$$

where S is the surface of the hemisphere $x^2 + y^2 + z^2 = 1,\ z \geq 0$. (**Hint**: use cylindrical rather than spherical polar coordinates to perform the surface integral).

(c) State Stokes' theorem and verify it by comparing your answers to parts (a) and (b). Explain how Stokes' theorem allows you to find the value of the following integral without further calculations,

$$\iint_{S'} \mathbf{G} \cdot d\mathbf{S},$$

where S' is the paraboloidal surface $z = 1 - x^2 - y^2$ for $0 \leq z \leq 1$ and $\mathbf{G} = (0, 0, 1 - x^2 - y^2)$.

END