MATH236001

This question paper consists of 4 printed pages, each of which is identified by the reference MATH2360

Only approved basic scientific calculators may be used

Examination for the Module MATH2360
(January 2004)

VECTOR CALCULUS AND APPLICATIONS

Time allowed: 2 hours

Do not attempt more than FOUR questions.
All questions carry equal weight.

1. **(i)** Show that the vector field

$$\boldsymbol{F}(x,y,z) = \left(-\frac{y}{x^2}\,\cos\left(\frac{y}{x}\right),\, \frac{1}{x}\,\cos\left(\frac{y}{x}\right),\, 2z\right)$$

obeys $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$, and find a corresponding potential field φ. What can you conclude about the line integral

$$\int_P^Q \boldsymbol{F} \cdot d\boldsymbol{r}\,?$$

Evaluate the integral over any arbitrary curve from $P = (1, \pi, 1)$ to $Q = (\frac{1}{2}, \pi/8, 2)$.

(ii) Consider the surface given by

$$\varphi(x,y,z) = z^3 - \exp(x + yz) = 0\ .$$

Find $\boldsymbol{\nabla}\varphi$ and calculate the unit normal vector to the surface at the point $(-1, 1, 1)$. Find the equation for the tangent plane and the shortest distance of this plane to the origin.

(iii) Sketch the region of integration for the integral

$$I = \int_0^1 \int_y^1 x^4\,\cos(y\,x^2)\,dx\,dy.$$

Evaluate I by interchanging the order of integration.

2. **(i)** Let $\boldsymbol{F}(x,y,z)$ be the vector field $\boldsymbol{F} = (z^{1/2}x, e^y, \sin z \cos x)$ and $\varphi(x,y,z)$ be the scalar field $\varphi = \sin(y+z)$.

Calculate $(\boldsymbol{F}\cdot\boldsymbol{\nabla})\varphi$ and $(\boldsymbol{F}\cdot\boldsymbol{\nabla})\boldsymbol{F}$.

(ii) Show, using suffix notation techniques or otherwise, that for an arbitrary vector field $\boldsymbol{v}$

$$\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{v}) = \boldsymbol{\nabla}\Big(\frac{1}{2}v^2\Big) - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$$

where $v^2 = \boldsymbol{v}\cdot\boldsymbol{v}$.

Hence, or otherwise, show that

$$\boldsymbol{\nabla}\times(\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{v})) + \boldsymbol{\nabla}\times((\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}) = 0 \qquad (*)$$

for an arbitrary vector field $\boldsymbol{v}$

Demonstrate the above result $(*)$ for the particular case where $\boldsymbol{v} = (z^2, x, 0)$.

3. **(i)** Calculate the integral

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r},$$

where $\boldsymbol{F} = (y\ , x^3\ , 0)$

in which the contour C is the circle $x^2+y^2 = 1$ in the plane $z = 0$ transversed in an anti-clockwise direction.

Hint: You may assume that

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \cos^4\theta = \frac{3}{8} + \frac{\cos 2\theta}{2} + \frac{\cos 4\theta}{8}$$

(ii) Evaluate

$$\iint_S (\boldsymbol{\nabla}\times\boldsymbol{F})\cdot d\boldsymbol{S},$$

where S is the surface given by $z^2 = 1-(x^2+y^2)$ $(z \geq 0)$ and $\boldsymbol{F}$ is the same as in **(i)**.

(iii) State Stokes' Theorem and verify it in this case by comparing the results of **(i)** and **(ii)**. Let I be defined by

$$I = \iint_S \boldsymbol{G}\cdot d\boldsymbol{S},$$

where S is the cone given by $z = 1-(x^2+y^2)^{1/2}$, $z \geq 0$ and $\boldsymbol{G} = (0\ , 0\ , 3x^2-1)$. State, using Stokes' Theorem, *and without any further calculations*, the value of I from the result of **(i)**.

Hint: Compare $\boldsymbol{G}$ with $\boldsymbol{\nabla}\times\boldsymbol{F}$.

4. **(i)** Calculate directly the surface integral

$$I = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

where $\boldsymbol{F} = (x^2\,,\, y\,,\, z)$ and S is the surface given by $z = 1 - x^4 - y^4 - 2x^2y^2$ with $0 \leq z \leq 1$.

Hint: Consider the expression $(x^2 + y^2)^2$ when considering the surface.

(ii) Calculate $\boldsymbol{\nabla} \cdot \boldsymbol{F}$, where $\boldsymbol{F}$ is defined as in **(i)**, and then calculate

$$J = \iiint_V (\boldsymbol{\nabla} \cdot \boldsymbol{F})\, dV$$

where V is the volume enclosed by the surface given in **(i)** and the disk $x^2 + y^2 = 1$, $z = 0$.

(iii) State the Divergence theorem and compare the results of questions **(i)** and **(ii)** to show that it is verified in the above case.

Hint: Remember to include the contribution from the disk.

5. **(i)** Calculate

$$\int_C f(x, y, z)\, ds$$

where s is the arc length parameter, $f(x, y, z) = \dfrac{zy^3}{2} + \dfrac{x}{2}\sin\left(\dfrac{\pi z}{2}\right)$, and C is the curve given by $x = t^3$, $y = t$, $z = 1$, with $0 \leq t \leq 1$.

(ii) By transforming to polar coordinates (r, θ) find

$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sin(x^2 + y^2)\, dx\, dy,$$

and sketch the region of integration in the (x, y)-plane.

(iii) Elliptic cylinder coordinates (u, v, w) are related to Cartesian coordinates (x, y, z) via the relations:

$$\begin{cases} x &= a\cosh u \cos v \\ y &= a \sinh u \sin v \\ z &= w \end{cases},$$

where $-\infty \leq u, v, w < \infty$.

Find the expressions for the scalars h_1, h_2, h_3 and the unit vectors $\mathbf{e}_u$, $\mathbf{e}_v$, $\mathbf{e}_w$ obeying

$$\frac{\partial \mathbf{x}}{\partial u} = h_1 \mathbf{e}_u \;\;,\;\; \frac{\partial \mathbf{x}}{\partial v} = h_2 \mathbf{e}_v \;\;,\;\; \frac{\partial \mathbf{x}}{\partial w} = h_3 \mathbf{e}_w \;,$$

where $\mathbf{x} = \mathbf{x}(u, v, w)$ is the position vector in the Cartesian coordinate system.

Show that the vectors $\mathbf{e}_u$, $\mathbf{e}_v$, $\mathbf{e}_w$ are orthonormal (orthogonal and of unit length).

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