**MATH236001**

This question paper consists of 3 printed pages, each of which is identified by the reference MATH2360

Only approved basic scientific calculators may be used



**Examination for the Module MATH2360**
**(January 2002)**

**VECTOR CALCULUS AND APPLICATIONS**

**Time allowed: 2 hours**

**Do not attempt more than FOUR questions.**
**All questions carry equal weight.**

**1.** **(i)** Show that the vector field

$$\boldsymbol{F}(x,y,z) = \left(y + e^{x+z^2}\,,\, x\,,\, 2ze^{x+z^2}\right)$$

obeys $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$, and find a corresponding potential field $\varphi$, such that $\boldsymbol{F} = \boldsymbol{\nabla}\varphi$. What can you conclude about the line integral

$$\int_P^Q \boldsymbol{F} \cdot d\boldsymbol{r}\,?$$

Evaluate the integral over an arbitrary curve from $P = (1, 1, \sqrt{2})$ to $Q = (1, 2, \sqrt{3})$.

**(ii)** Consider the surface given by

$$\varphi(x,y,z) = x^2 \ln x + y^{3/2} + z - 1 = 0\ .$$

Find $\boldsymbol{\nabla}\varphi$ and calculate the unit normal vector to the surface at the point $(1, 4, 3)$.

**(iii)** Sketch the region of integration for the integral

$$I = \int_0^1 \int_{x^2}^1 x^3 \,\left(1 + y^3\right)^{1/5}\, dy\, dx.$$

Evaluate $I$ by interchanging the order of integration.

**2.** **(i)** Let $\boldsymbol{F}(x,y,z)$ be the vector field $\boldsymbol{F} = (y \sin x\,,\, \cos^2 z\,,\, x^{3/2})$ and $\varphi(x,y,z)$ be the scalar field $\varphi = (x-y)^2 + z$.

Calculate $\boldsymbol{\nabla}\varphi$, $\boldsymbol{\nabla} \times \boldsymbol{F}$, $(\boldsymbol{F} \cdot \boldsymbol{\nabla})\varphi$ and $(\boldsymbol{F} \cdot \boldsymbol{\nabla})\boldsymbol{F}$.

**(ii)** Show, using suffix notation techniques or otherwise, that for an arbitrary vector field $\boldsymbol{G}$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{G}) = \boldsymbol{\nabla}\,(\boldsymbol{\nabla} \cdot \boldsymbol{G}) - \nabla^2 \boldsymbol{G}.$$

Verify this formula, taking $\boldsymbol{G}$ to be the vector $\boldsymbol{F}$ of part **(i)**.

**3.** **(i)** Calculate the integral

$$\oint_C \boldsymbol{v} \cdot d\boldsymbol{r},$$

where $\boldsymbol{v} = (x^2 - z^2\ , x^3 - zy\ , xy^2)$, in which the contour $C$ is the circle $x^2 + y^2 = 1$ in the plane $z = 0$ traversed in an anti-clockwise direction.

**Hint:** You may assume that

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \cos^4\theta = \frac{3}{8} + \frac{\cos 2\theta}{2} + \frac{\cos 4\theta}{8}.$$

**(ii)** Evaluate

$$\iint_S (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot d\boldsymbol{S},$$

where $S$ is the conical surface given by $(x^2 + y^2)^{1/2} + z = 1$ $(0 \le z \le 1)$ and $\boldsymbol{v}$ is the same as in **(i)**.

**(iii)** State Stokes' Theorem and verify it in this case by comparing the results of **(i)** and **(ii)**.

**4.** **(i)** Calculate directly the surface integral

$$I = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S},$$

where $\boldsymbol{F} = (x - y\ , y + x\ , x^2 + y^2)$ and $S$ is the paraboloid given by $x^2 + y^2 + z = 4$, with $0 \le x \le 2$, $0 \le y \le 2$, $0 \le z \le 4$.

**(ii)** Calculate $\boldsymbol{\nabla} \cdot \boldsymbol{F}$, where $\boldsymbol{F}$ is defined as in **(i)**, and then calculate

$$J = \iiint_V (\boldsymbol{\nabla} \cdot \boldsymbol{F})\, dV,$$

where $V$ is the volume enclosed by the paraboloid in section **(i)** and the disk given by $x^2+y^2 \le 4$, $z = 0$.

**(iii)** State the Divergence theorem and compare the results of questions (i) and (ii) to show that it is verified in the above case.

**Hint:** Remember to add in the contribution from the disk in the $xy$-plane, which together with the paraboloid, forms the closed surface for the volume in part **(ii)**.

**5.** **(i)** Calculate

$$\oint_C f(x, y, z)\, ds,$$

where $s$ is the arc length parameter, $f((x, y, z) = x\, y^{1/2}\, (1 - 2xz) + 2\, (1 - 2x^2)^{1/4}\, x^3$, and $C$ is the closed curve given by $x = \dfrac{\sin t}{\sqrt{2}}$, $y = \cos t$ and $z = \dfrac{\sin t}{\sqrt{2}}$.

**(ii)** By means of the transformation $x = ar\cos\theta$, $y = br\sin\theta$, evaluate

$$\iint_{\mathcal{R}} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{5/4} dy\, dx,$$

where $\mathcal{R}$ denotes the region bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$.

**(iii)** Oblate spheroidal coordinates $(\xi, \eta, \phi)$ are related to Cartesian coordinates $(x, y, z)$ via the relations

$$\begin{aligned} x &= a \cosh \xi \cos \eta \cos \phi, \\ y &= a \cosh \xi \cos \eta \sin \phi, \\ z &= a \sinh \xi \sin \eta, \end{aligned}$$

where $a$ is a constant and $\xi \geq 0$, $-\frac{\pi}{2} \leq \eta \leq \frac{\pi}{2}$, $0 \leq \phi < 2\pi$.

Find the expressions for the scalars $h_1, h_2, h_3$ and the unit vectors $\mathbf{e}_\xi$, $\mathbf{e}_\eta$, $\mathbf{e}_\phi$ obeying

$$\frac{\partial \mathbf{x}}{\partial \xi} = h_1 \mathbf{e}_\xi \,, \quad \frac{\partial \mathbf{x}}{\partial \eta} = h_2 \mathbf{e}_\eta \,, \quad \frac{\partial \mathbf{x}}{\partial \phi} = h_3 \mathbf{e}_\phi \,,$$

where $\mathbf{x} = \mathbf{x}(\xi, \eta, \phi)$ is the position vector in the Cartesian coordinate system.

Show that the vectors $\mathbf{e}_\xi$, $\mathbf{e}_\eta$, $\mathbf{e}_\phi$ are orthogonal.

**End**