MATH–201101

This question paper consists of 2 printed pages, each of which is identified by the reference MATH–201101

Only approved basic scientific calculators may be used.



Examination for the Module MATH–2011

(January 2004)

**Real Analysis 2**

Time allowed: 2 hours

Answer **four** questions. All questions carry equal marks.

**1. (a)** State Taylor's Theorem, giving both Lagrange's and Cauchy's forms of the remainder.

Use Taylor's Theorem for the function $\sin(\pi e^x)$ to prove that if $x \geqslant 0$ then

$$|\sin(\pi e^x) + \pi x| \leqslant \tfrac{1}{2}x^2 \pi e^x (1 + \pi e^x).$$

**(b)** Use Taylor's Theorem to express $x^3 + 5x^2 - 2x + 4$ as a polynomial in $x + 4$.

**2. (a)** Determine whether or not each of the following sequences has a convergent subsequence. Give reasons for your answers.

$(i)$ $x_n = \cos \frac{n\pi}{5} + \sin \frac{n\pi}{7}$,

$(ii)$ $x_n = \left(\frac{3}{4}\right)^n - \left(\frac{4}{3}\right)^n$,

$(iii)$ $x_n = n^2 \left(\cos \frac{n\pi}{4} - \sin \frac{n\pi}{4}\right)$.

**(b)** Give the definition of a *Cauchy sequence.* Prove from your definition that every subsequence of a Cauchy sequence is a Cauchy sequence.

Prove also that if a Cauchy sequence $(x_n)$ has a subsequence that converges to a limit $\ell$, then $(x_n)$ converges to $\ell$.

**3. (a)** Define what is meant by saying that the function $f : J \to \mathbb{R}$ is *uniformly continuous* on the interval $J \subseteq \mathbb{R}$. Prove that if $f$ has a bounded derivative on $J$, say $|f'(x)| \leqslant M$ for all $x$ in $J$, then $f$ is uniformly continuous on $J$.



**Question 3 continues . . .**

**(b)** Find the upper and lower Riemann sums for the function $f(x) = 3 - 2x^2$ on $[1, 2]$ for a dissection of the interval into $n$ equal subintervals. Hence find the value of $\int_1^2 (3 - 2x^2)\, dx$.
[You may assume that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.]

**4. (a)** State both the first and the second forms of the Fundamental Theorem of the Calculus. Prove the second form, assuming the truth of the first.

**(b)** Find $\dfrac{d}{dx}\displaystyle\int_x^{x^2} \cos\Big(\frac{\pi}{t}\Big)\, dt$, for $x > 0$.

**(c)** Find which of the following improper integrals converge, and evaluate any that do:

$$(i) \quad \int_1^{\infty} \frac{1}{x^2 + 3x}\, dx, \qquad (ii) \quad \int_1^{\infty} \frac{1}{1 + \sqrt{x}}\, dx.$$

**5. (a)** Explain what is meant by saying that a sequence of functions $(f_n)$ *converges uniformly* to a limit function $f$ on a set $J$. Prove that if each $f_n$ is continuous then $f$ is also continuous.

**(b)** Define sequences of functions $(f_n)$, $(g_n)$ by $f_n(x) = e^{-nx} \cos x$, $g_n(x) = e^{-nx} \sin x$. By finding its pointwise limit, deduce from part **(a)** that $(f_n)$ does not converge uniformly on $[0, \pi/2]$. Show however that $(g_n)$ does converge uniformly on $[0, \pi/2]$.

**(c)** Use the power series expansion for $e^x$ to show that

$$\int_0^a e^{-t^2}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{n!(2n+1)},$$

indicating any theorems on uniform convergence that you use.

**END**