MATH–201101

This question paper consists of 2 printed pages, each of which is identified by the reference MATH–201101

Only approved basic scientific calculators may be used.

Examination for the Module MATH–2011

(January 2003)

Real Analysis 2

Time allowed: 2 hours

Answer **four** questions. All questions carry equal marks.

1. **(a)** State Taylor's Theorem with Lagrange's form of the remainder.

Use Taylor's Theorem for the function $(x-1)\cos x$ to prove that if $0 \leqslant x \leqslant \pi$ then

$$|(x-1)\cos x + 1 - x - \tfrac{1}{2}x^2| \leqslant \frac{2+\pi}{6}|x|^3.$$

(b) Write down the binomial series for the function $(1+x)^{1/3}$, giving an expression for the general term in the series. Estimate the error in using just the first four terms of this series to evaluate the cube root of 1.25.

2. **(a)** Give the definition of a *Cauchy sequence*.

Prove that every Cauchy sequence (x_n) of real numbers converges. [You may assume the Bolzano–Weierstrass theorem, namely that every bounded sequence contains a convergent subsequence.]

(b) State, giving brief reasons, whether (x_n) is a Cauchy sequence in each of the following cases:

$$(i) \quad x_n = \ln n, \qquad (ii) \quad x_n = \frac{\ln n}{n}, \qquad (iii) \quad x_n = n \sin\frac{1}{n}.$$

(c) Give a careful definition of a *rearrangement* of a series.

State (without proof) Dirichlet's theorem about rearrangements of an absolutely convergent series.

3. **(a)** Show that the function $f(x) = 2x^2 - x$ is uniformly continuous on the interval $[-3, 3]$ by determining, for each $\varepsilon > 0$, a $\delta > 0$ such that

$$|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon \text{ for all } x,\ y \in [-3, 3].$$

Show that the function $h(t) = 1/t \quad (0 < t \leqslant 1)$ is *not* uniformly continuous on $(0, 1]$.

(b) Find the upper and lower Riemann sums for the function $f(x) = 10 - 3x$ on $[2, 4]$ for a dissection of the interval into n equal subintervals of length h, and hence find the value of $\int_2^4 (10 - 3x)\, dx$.

4. **(a)** Calculate the derivative of the function $F(x) = \int_{x^2}^{x^3} \ln\left(1 + \frac{1}{2}\sin\theta\right) d\theta$.

(b) State carefully (but without proof) the Integral Test for convergence of series.

(c) Determine (giving reasons) whether each of the following improper integrals is convergent or divergent:

$$(i) \quad \int_1^\infty \frac{x^3}{1 + x^4}\, dx, \qquad (ii) \quad \int_0^\infty e^{-x^2}\, dx, \qquad (iii) \quad \int_0^{\pi/2} (\sin x)^{-1/2}\, dx.$$

5. **(a)** Explain what is meant by saying that a sequence of functions (f_n) *converges uniformly* to a limit function f on a set X.

Determine whether each of the following sequences of functions (f_n), (g_n) converges uniformly to the zero function on $\mathbb{R}$:

$$(i) \quad f_n(x) = \frac{x}{1 + nx^2}, \qquad (ii) \quad g_n(x) = \frac{x}{1 + n^2}.$$

(b) Find the pointwise sum of the (geometric) series $\sum_{n=1}^{\infty} \frac{x^2}{(1 + x^2)^n}$ for $0 \leqslant x \leqslant 1$. Explain how this shows that $\sum_{n=1}^{\infty} \frac{x^2}{(1 + x^2)^n}$ is *not* uniformly convergent on $[-1, 1]$.

END