MATH–201101

This question paper consists of 2 printed pages, each of which is identified by the reference MATH–2011

Only approved basic scientific calculators may be used.

Examination for the Module MATH–2011

(January 2002)

Real Analysis 2

Time allowed : 2 hours

Answer **four** questions. All questions carry equal marks.

1. **(a)** State Taylor's Theorem, giving both Lagrange's and Cauchy's form of the remainder.

Using Taylor's Theorem with Lagrange remainder for the function $f(x) = (9+x)^{-\frac{1}{2}}$, prove that if $|x| < 5$ then $(9+x)^{-\frac{1}{2}} - \frac{1}{3} + \frac{x}{54}$ lies between 0 and $\frac{75}{256}$.

(b) Use Taylor's Theorem to express $x^4 - 2x^3 - x^2 + x + 3$ as a polynomial in $x - 2$.

2. **(a)** Give the definition of a *Cauchy sequence*.

Prove that every convergent sequence (x_n) of real numbers is a Cauchy sequence.

State, giving brief reasons, whether (x_n) is a Cauchy sequence in each of the following cases:

$$(i) \quad x_n = \sqrt{n}, \qquad (ii) \quad x_n = \frac{\sqrt{1+n^2}}{1+n}.$$

(b) What is meant by a *conditionally convergent series*?

Give a careful definition of a *rearrangement* of a series.

State a theorem about the possible values of the sum of a rearrangement of a conditionally convergent series. Give a short description (not a formal proof) of how the theorem is proved.

3. **(a)** Define the *upper* and *lower Riemann sums* of a bounded function f on a closed interval $[a, b]$ corresponding to a dissection $\mathcal{D}$ of $[a, b]$ at the points $a = a_0 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n = b$. What is meant by saying that f is *Riemann integrable* on $[a, b]$?

Give (with a brief justification) examples of functions f and g such that

(i) f is Riemann integrable on $[a, b]$ but is not continuous,

(ii) g is bounded but not Riemann integrable on $[a, b]$.

(b) Find the upper and lower Riemann sums for the function $f(x) = 3 - 4x^2$ on $[0, 1]$ for a dissection of the interval into n equal subintervals of length h, and hence find the value of $\int_0^1 (3 - 4x^2)\,dx$. [You may assume that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.]

4. (a) Prove the following form of the Fundamental Theorem of Calculus: if $f : I \to \mathbb{R}$ is a continuous function on an open interval I, $a \in I$ and $F(x) = \int_a^x f(t)\,dt$ then F is differentiable on I with derivative equal to f.

(b) Find $\dfrac{d}{dx}\displaystyle\int_1^{e^{x^2}} \frac{1}{1+\log t}\,dt$.

(c) Use the comparison test to determine whether or not the improper integral $\displaystyle\int_0^{\pi/2} \frac{1}{\sin x}\,dx$ converges.

5. (a) Explain what is meant by saying that a sequence of functions (f_n) *converges uniformly* to a limit function f on a set X.

Determine whether each of the following sequences of functions (f_n), (g_n) converges uniformly on the interval $[0, 1]$:

$$(i) \quad f_n(x) = x^n(1 - x^n), \qquad (ii) \quad g_n(x) = \sqrt{n}\,x e^{-nx}.$$

(b) State the Weierstrass M-test. Use it to prove that that the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^2}{n(1+nx^2)}$ converges uniformly on $\mathbb{R}$.

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