MATH1970-01

This question paper consists of 2 printed pages, each of which is identified by the reference MATH1970-01

Only approved basic scientific calculators may be used

Examination for the Module MATH1970
(May 2004)

Differential Equations

Time allowed: 2 hours

Attempt no more than 4 questions.

You must show your working in answers to all questions.

1. (a) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} - \pi y = 0\,,$$

with the initial condition $y(0) = \frac{1}{2}$. Sketch the solution.

(b) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} - \pi y = -\frac{\pi}{2}\,,$$

with the initial condition $y(0) = 1$. Sketch the solution.

(c) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} - \pi y = -y^3\,,$$

with the initial condition $y(0) = \sqrt{\pi/2}$. Sketch the solution.

2. (a) Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} - 2ty = t.$$

(b) Find the general solution of the differential equation

$$\frac{2y(x-1)}{y^2+1}\frac{\mathrm{d}y}{\mathrm{d}x} + \ln\left(y^2+1\right) = 0\,.$$

(c) Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \frac{1}{3}\left(\frac{x}{y}\right)^2\,.$$

3. (a) Find the general solution of the differential equation

$$y'' - y' + \frac{1}{4}y = 0.$$

Hint: Check that you have two linearly independent solutions by calculating the Wronskian, $W(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$.

(b) Three solutions of a certain second-order inhomogeneous linear differential equation are

$$y_{\rm i}(t) = 1 + {\rm e}^{t^2}\,, \qquad y_{\rm ii}(t) = 1 + t{\rm e}^{t^2}\,, \qquad y_{\rm iii}(t) = (t+1){\rm e}^{t^2} + 1\,.$$

Find the general solution of the differential equation.

The differential equation is of the form

$$y'' + p(t)y' + q(t)y = g(t).$$

Find $p(t)$, $q(t)$ and $g(t)$.

Hint: the Wronskian is related to $p(t)$ *by* $W(t) = A\exp(-\int p(t){\rm d}t)$.

4. (a) Find the general solution of the differential equation

$$m\ddot{x} = -kx\ . \tag{1}$$

(b) Find the solution of (1) with the initial conditions $x(0) = -U$, $\dot{x}(0) = V$, where $U > 0$ and $V > 0$.

Write your solution in the form

$$x(t) = A\cos(\omega t - \phi).$$

Find A, ω and ϕ in terms of U and V. Sketch $x(t)$ and $\dot{x}(t)$.

(c) Find a particular solution of the differential equation

$$\ddot{x} + x = \cos x\,.$$

5. The set of equations

$$\begin{aligned} x_1' &= \alpha x_1 + 3x_2 \\ x_2' &= -3x_1\ , \end{aligned}$$

can be written as one equation by defining $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then

$$\mathbf{x}' = \begin{pmatrix} \alpha & 3 \\ -3 & 0 \end{pmatrix}\mathbf{x}\,. \tag{2}$$

The eigenvalues of the 2×2 matrix in (2) satisfy $\lambda^2 - \alpha\lambda + 9 = 0$.

(a) What type of fixed point is the origin (stable node, unstable node, saddle, centre, stable spiral or unstable spiral) if $\alpha = 5$?

(b) What type of fixed point is the origin if $\alpha = 10$?

(c) Sketch the trajectory for $\alpha = 0$ with the initial condition $x_1 = 1$, $x_2 = 0$.

(d) Sketch the trajectory for $\alpha = 10$ with the initial condition $x_1 = 1$, $x_2 = 0$.

What is the value of $\dfrac{x_2}{x_1}$ as $t \to \infty$?

END