MATH196001

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH196001**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH1960

(January 2007)

CALCULUS

Time allowed: **2 hours**

Attempt all six questions from Section A and no more than three questions from Section B.

Section A is worth 40% and Section B 60% of the available marks.

SECTION A

A1. Differentiate with respect to x the functions

(a) $\cos\left((x-1)^3\right)$,

(b) $\dfrac{\ln(x^2+1)}{x+2}$.

A2. Starting from the definitions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \ \sinh x = \frac{e^x - e^{-x}}{2}$$

of the basic hyperbolic functions, prove the identity $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$.

A3. Using the definition $\tanh x = \dfrac{(e^x - e^{-x})}{(e^x + e^{-x})}$, show that

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right).$$

Hence, or otherwise, obtain the derivative of $\tanh^{-1} x$.

A4. Obtain the derivative of the function $f(x) = x^{2x}$ with respect to x.

A5. Consider the function $f(x, y) = x^2y - 2y^2$.

(a) Find $\underline{\nabla} f$.

(b) At $P(1, -1)$ find the rate of change of f in the direction of the vector $\underline{v} = 4\underline{i} + 3\underline{j}$.

A6. Use integration by parts to find the indefinite integral

$$\int x \sinh x dx.$$

Hence, show that $\int_0^{\ln 2} x \sinh x dx = \frac{1}{4}(5\ln 2 - 3)$.

SECTION B

B1. The function $f(x)$ is defined as

$$f(x) = \frac{x^2 \exp(x)}{(x-1)^2}.$$

(a) Find and classify the stationary points of $f(x)$.
[Hint: consider whether the first derivative is increasing or decreasing through each stationary point].

(b) Sketch a graph of $f(x)$, indicating stationary points, axis crossings and horizontal and vertical asymptotes.

(c) Find, or write down, the first three non zero terms of the Taylor series for the functions $g(x) = \exp(x)$ and $h(x) = (x-1)^{-2}$ about the point $x = 0$. Hence, or otherwise, find the first three non zero terms of the Taylor series for $f(x)$ about $x = 0$.

B2. (a) Find the gradient $\underline{\nabla} f$ of the function $f(x, y) = x^3 + 3y^3 + 3x^2y - 3xy^2 - 3x - 3y$ and show that $f_y - f_x = 12y(y - x)$.

(b) Find the tangent plane to the surface $z = f(x, y)$ at the point $(1, 1)$, giving your answer in the form $ax + by + cz = d$.

(c) Find the second-order partial derivatives f_{xx}, f_{xy} and f_{yy}.

(d) Find and classify the stationary points of $f(x, y)$, giving the values of x, y and f at each point.

B3. (a) Given the co-ordinate transformation

$$x = s^2 - t^2, \qquad y = s - t$$

calculate the Jacobean of the transformation

$$\frac{\partial (x, y)}{\partial (s, t)}$$

and, if possible, express s and t in terms of x and y.

(b) Using the extended chain rule for the above transformation, show that, for an arbitrary function $f(x, y)$,

$$tf_s + sf_t = -yf_y.$$

(c) Find the distances to the nearest and furthest points from the origin which lie on the elipse

$$(x + 3)^2 + 4y^2 = 32.$$

B4. (a) Use the substitution $t = x^2 - 2x + 5$ to find the indefinite integral

$$\int \frac{x - 1}{x^2 - 2x + 5} dx.$$

(b) Use the substitution $x = 2 \tan u + 1$ to find the indefinite integral

$$\int \frac{1}{x^2 - 2x + 5} dx.$$

(c) Using partial fractions, and your answers to parts (a) and (b), show that

$$\int_1^3 \frac{3x^2 - 5x}{x^3 - x^2 + 3x + 5} dx = 2 \ln 2 - \frac{3\pi}{8}.$$

[Hint: Note that $x = -1$ is a solution of $x^3 - x^2 + 3x + 5 = 0$.]

(d) Use integration by parts to obtain the reduction formula,

$$I_n = x (\ln x)^n - nI_{n-1}$$

for the integral

$$I_n = \int (\ln x)^n \, dx.$$

Hence, or otherwise, evaluate $\int_1^e (\ln x)^3 \, dx$.

END