MATH196001

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH196001**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH1960
(January 2006)

CALCULUS

Time allowed: **2 hours**

Attempt all six questions from Section A and
no more than three questions from Section B.

Section A is worth 40% and Section B 60% of the available marks.

SECTION A

A1. Differentiate with respect to x the functions

(a) $\sin\left(x^4\right)$,

(b) $x^3\ln(5x)$.

A2. Starting from the definitions

$$\cosh x = \frac{e^x + e^{-x}}{2},\ \sinh x = \frac{e^x - e^{-x}}{2}$$

of the basic hyperbolic functions, prove the identity $\sinh(x-y) = \sinh x\cosh y - \cosh x\sinh y$.

A3. Using the result $\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$, show that, if $y = \sin^{-1} x$, then

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

A4. Obtain the first two derivatives of the function $f(x) = x^{-1}$ and hence obtain the first three terms of its Taylor series about $x = 1$.

A5. Find the tangent plane to the surface $z = x^2 + y^3$ at the point $x = 2$, $y = 1$, giving your answer in the form $ax + by + cz = d$.

A6. Use integration by parts to find the indefinite integral

$$\int x \sin x dx.$$

Hence, find the value of $\int_0^\pi x \sin x dx$.

SECTION B

B1. **(a)** Show that the first derivative of the function

$$f(x) = \frac{2x-1}{x-1} \exp(x)$$

can be written in the form

$$\frac{df}{dx} = \frac{ax^2 - bx}{(x-1)^2} \exp(x)$$

where a and b are numbers which you are required to find.

(b) Find and classify the stationary points of $f(x)$.

[Hint: consider the sign of the first derivative on either side of each stationary point].

(c) Sketch a graph of $f(x)$, indicating stationary points, axis crossings and horizontal and vertical asymptotes.

B2. **(a)** Find the gradient of the function $f(x, y) = 4xy - x^2 - 4y^2 + \exp(x^3 - 3x^2)$.

(b) Find the second-order partial derivatives f_{xy} and f_{yy}, and show that

$$f_{xx} = -2 + \left(6x - 6 + 9\left(x^2 - 2x\right)^2\right) \exp\left(x^3 - 3x^2\right)$$

(c) Find and classify the stationary points of $f(x, y)$, giving the values of x, y and f at each point.

B3. **(a)** The volume of a cone of base radius x and height y is given by

$$V = \frac{\pi}{3} f(x, y) \quad \text{where} \quad f(x, y) = x^2 y,$$

and the area of its curved surface is

$$A = \pi g(x, y) \quad \text{where} \quad g(x, y) = x\left(x^2 + y^2\right)^{\frac{1}{2}}.$$

Find the maximum volume of a cone subject to the constraint of a fixed area A (i.e. $g(x, y) =$ constant) and find the values of x and y which correspond to this maximum volume. Express your answers in terms of the area, A.

(b) For the curve defined by $x(x^2 + y^2)^{\frac{1}{2}} = 1$ show that

$$\frac{dy}{dx} = -\frac{2x^2 + y^2}{xy}.$$

B4. **(a)** Use the substitution $u = x^2 + 1$ to evaluate

$$\int_0^1 \frac{x^5}{x^2 + 1} dx.$$

(b) Using partial fractions, find the indefinite integral

$$\int \frac{5x - 1}{x^3 - 2x^2 - x + 2} dx.$$

[Hint: Note that $x = 1$ is a solution of $x^3 - 2x^2 - x + 2 = 0$.]

END