MATH171501

This question paper consists of 5 printed pages, each of which is identified by the reference **MATH171501**.

Statistical tables are attached at the end of the question paper. Only approved basic scientific calculators may be used.

Examination for the Module MATH1715
(January 2006)

INTRODUCTION TO PROBABILITY

Time allowed: **2 hours**

Attempt ALL questions in Section A and THREE questions from Section B.

For Section A, only write down a single letter answer for each question.

Section A is worth 40% of the examinations marks.

All questions within each section carry equal marks.

Section A

Attempt ALL questions in Section A.

For each question, write down a single letter answer.

A1. If events A and B have probabilities 0.7 and 0.3, respectively, and $\mathsf{P}(A \cup B) = 0.8$, what is the probability that both events A and B occur?

A: 0.4 B: 0.21 C: 0.2 D: 0.8 E: 1

A2. A computer company provides an insurance policy for one of its systems. If the system fails during the first year, the policy pays £3,000. The benefit decreases by £1,000 each year until it reaches £0. If the system has not failed at the beginning of a year, the probability that it fails during that year is 0.1. How much must the company charge for the insurance policy so that, on average, its net gain per policy is £100?

A: £570 B: £500 C: £561 D: £700 E: £661

A3. Let X be a geometrically distributed random variable with parameter $p = 0.8$. Set $Y = \min\{X, 2\}$, so that $Y = X$ when $X \le 2$ and $Y = 2$ otherwise. What is the expected value of Y?

A: 1.20 B: 1.25 C: 1.12 D: 4.00 E: 5.00

A4. Suppose that the infant mortality rate is 5 per 1000 births. Use the Poisson approximation to evaluate the probability that there are at most 2 infant deaths out of 400 births.

A: 0.010 B: 0.406 C: 0.677 D: 0.323 E: 0.594

A5. In a small lottery, 10 tickets are sold (numbered $1, 2, \ldots, 10$). Two lucky numbers are drawn at random for prizes. You hold two tickets numbered 1 and 4. What is the probability that you win at least one prize?

A: $\frac{1}{5}$ B: $\frac{3}{10}$ C: $\frac{9}{25}$ D: $\frac{17}{45}$ E: $\frac{28}{45}$

A6. Five people play a game of "odd man out" to determine who will pay for the pizza they ordered. Each flips a coin. If only one person gets heads (or tails) while the other four get tails (or heads), then he is the odd man and has to pay. Otherwise they flip again. What is the expected number of tosses needed to determine who will pay?

A: 3.0 B: 3.2 C: 2.0 D: 2.5 E: 1.6

A7. The number of emergency patients arriving to a hospital during a typical Friday has a Poisson distribution with parameter $\lambda = 5.9$. What is the probability that, on a given Friday, at least two patients will arrive?

A: 0.997 B: 0.981 C: 0.984 D: 0.508 E: 0.661

A8. Suppose that the lifetime of a light bulb produced by a certain company is a normal random variable with mean 1000 hours and standard deviation 100 hours. What percentage of light bulbs produced by that company will last at least 900 hours?

A: 15.87% B: 30.85% C: 31.74% D: 68.26% E: 84.13%

A9. A balanced die is rolled 180 times. Let X be the number of times where the die shows a “6”. Use the normal approximation with continuity correction to evaluate the probability that $X = 30$.

A: 0.0398 B: 0.0796 C: 0.5000 D: 0.0056 E: 0.1192

A10. According to an opinion poll in a certain country, 54% of men and 33% of women believe in aliens from outer space. The governmental statistics shows that 48% of adults are men. What percentage of adult population believe in aliens?

A: 28.71% B: 43.50% C: 43.92% D: 43.08% E: 41.76%

Section B

Attempt THREE questions from Section B.

B1. (a) Let A, B, and C be arbitrary events. Determine if the following equality is correct:

$$(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C).$$

Illustrate your answer using Venn diagrams.

(b) Suppose that 40% of people in a certain town subscribe to newspaper A, 60% to newspaper B, and 30% do not subscribe to either of the papers. If someone subscribes to newspaper A, what is the probability that this person also subscribes to newspaper B?

(c) Suppose that, on average, 1% of a certain brand of Christmas light bulbs are defective. Compute the probability that in a box of 25 there will be at most one defective bulb. Use the Poisson approximation to compute the same probability, and briefly explain whether a close match could be anticipated. (Give the answers to 4 decimal places.)

B2. (a) If events A and B are independent, are the events A^c and B independent as well? Explain your answer carefully by referring to the definition of independent events.

(b) For a discrete random variable X with probability mass function $p_X(x)$, state the rule to compute the expected value of $Y = g(X)$, where $y = g(x)$ is a given function.

(c) Let X be a number picked at random according to the uniform distribution on $[0, 1]$, and set $Y = \sqrt{X}$.

(i) Evaluate the mean of Y.

(ii) Let Y_1 be the first digit in a decimal expansion of Y. Show that $Y_1 = 0$ with probability 0.01. Obtain the probability distribution of Y_1 and compute $\mathsf{E}(Y_1)$.

B3. (a) As you know, $A \cup B$ is the event that at least one of events A and B occurs. Using set notation, express the event that exactly one of A and B occurs, and draw a Venn diagram to represent this event.

(b) A blood test for hepatitis is 90% effective in detecting the disease when it is in fact present. However, the test yields a positive result for 1% of healthy persons tested. The disease rate in the general population is 1 in 10000.

(i) What is the probability that a person who receives a positive test result actually has hepatitis?

(ii) A patient is sent for a blood test because he has lost his appetite and has developed jaundice, that is, yellowness of the skin and of the sclera (the white of the eye). The physician knows that this type of patient will have hepatitis with probability 0.5. If this patient gets a positive result on his blood test, what is the probability that he has hepatitis?

Give your answers to 3 significant figures.

(c) Suppose a balanced coin is tossed repeatedly, and let T denote the number of tosses until the first head is obtained. Determine the probability that T is an odd number.

B4. (a) Suppose that a random variable X has a Poisson distribution with parameter λ. Show that its probability generating function is given by $G_X(s) = e^{-\lambda + \lambda s}$. Using this or otherwise, obtain the expected value of X.

(b) Let X and Y be independent random variables such that $\mathsf{E}(X) = 0$, $\mathsf{E}(Y) = 1$, $\mathsf{Var}(X) = 2$, and $\mathsf{Var}(Y) = 1$. Consider a new random variable, $Z = X - 2Y$. Find the variance of Z and the covariance $\mathsf{Cov}(X, Z)$.

(c) A continuous random variable X has probability density function given by

$$f_X(x) = \begin{cases} cx & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Determine the constant c and obtain the mean of X.

(ii) For the random variable $Y = X^2$, obtain its cumulative distribution function $F_Y(x)$ and probability density function $f_Y(x)$.

Normal Distribution Function Tables

The first table gives

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy$$

and this corresponds to the shaded area in the figure to the right. $\Phi(x)$ is the probability that a random variable, normally distributed with zero mean and unit variance, will be less than or equal to x. When $x < 0$ use $\Phi(x) = 1 - \Phi(-x)$, as the normal distribution with mean zero is symmetric about zero. For interpolation use the formula

$$\Phi(x) \approx \Phi(x_1) + \frac{x - x_1}{x_2 - x_1} \left(\Phi(x_2) - \Phi(x_1)\right)$$

$$(x_1 < x < x_2)$$

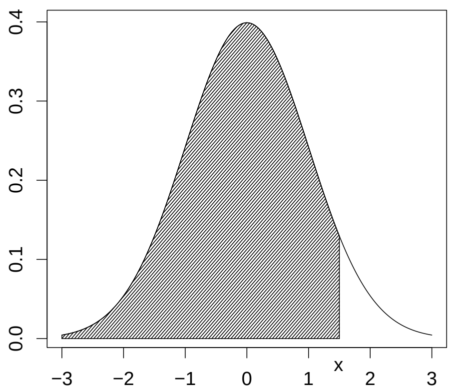

Table 1

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.00	0.5000	**0.50**	0.6915	**1.00**	0.8413	**1.50**	0.9332	**2.00**	0.9772	**2.50**	0.9938
0.05	0.5199	**0.55**	0.7088	**1.05**	0.8531	**1.55**	0.9394	**2.05**	0.9798	**2.55**	0.9946
0.10	0.5398	**0.60**	0.7257	**1.10**	0.8643	**1.60**	0.9452	**2.10**	0.9821	**2.60**	0.9953
0.15	0.5596	**0.65**	0.7422	**1.15**	0.8749	**1.65**	0.9505	**2.15**	0.9842	**2.65**	0.9960
0.20	0.5793	**0.70**	0.7580	**1.20**	0.8849	**1.70**	0.9554	**2.20**	0.9861	**2.70**	0.9965
0.25	0.5987	**0.75**	0.7734	**1.25**	0.8944	**1.75**	0.9599	**2.25**	0.9878	**2.75**	0.9970
0.30	0.6179	**0.80**	0.7881	**1.30**	0.9032	**1.80**	0.9641	**2.30**	0.9893	**2.80**	0.9974
0.35	0.6368	**0.85**	0.8023	**1.35**	0.9115	**1.85**	0.9678	**2.35**	0.9906	**2.85**	0.9978
0.40	0.6554	**0.90**	0.8159	**1.40**	0.9192	**1.90**	0.9713	**2.40**	0.9918	**2.90**	0.9981
0.45	0.6736	**0.95**	0.8289	**1.45**	0.9265	**1.95**	0.9744	**2.45**	0.9929	**2.95**	0.9984
0.50	0.6915	**1.00**	0.8413	**1.50**	0.9332	**2.00**	0.9772	**2.50**	0.9938	**3.00**	0.9987

The inverse function $\Phi^{-1}(p)$ is tabulated below for various values of p.

Table 2

p	**0.900**	**0.950**	**0.975**	**0.990**	**0.995**	**0.999**	**0.9995**
$\Phi^{-1}(p)$	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902	3.2905

END