MATH161001

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH1610**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH1610
(May/June 2007)

Discrete Systems

Time allowed: **2 hours**

Do not attempt more than **4** questions. All questions carry equal weight.

1. **(a)** If $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, find
(i) $\mathbf{a} - 2\mathbf{b}$,
(ii) $|\mathbf{a} + \mathbf{b}|$,
(iii) a unit vector in the direction of $2\mathbf{a} + \mathbf{b}$,
(iv) the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$,
(v) $\mathbf{a} \times \mathbf{b}$.

(b) Find the equation of the line L_1 through the point with position vector $(4, 2, 1)$ and parallel to the vector $(-3, 3, -9)$.
Find the equation of the line L_2 through the points with position vectors $(1, -1, 2)$ and $(3, 0, 3)$.
Show that the lines L_1 and L_2 intersect and find their point of intersection.

2. **(a)** Find a vector perpendicular to the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$.

(b) (i) Find the equation of the plane Π_1 through the point with position vector $(2, 1, 1)$ and perpendicular to the vector $(3, -1, 2)$.
Determine the perpendicular distance of the plane Π_1 from the origin.
(ii) Find the equation of the plane Π_2 which passes through the three points $(2, -2, 1)$, $(4, -1, 6)$ and $(3, -3, -2)$.
(iii) Find the angle between the planes Π_1 and Π_2.

3. (a) Show how the iterated map

$$x_{n+1} = \frac{x_n^2 + 6}{2x_n}, \qquad x_0 \neq 0 \text{ given}$$

can give $\sqrt{6}$.
Use this map to find $\sqrt{6}$, starting with $x_0 = 2.0$. You should show your working and give your result correct to 5 decimal places.

(b) Find the general solution to the non-homogeneous linear maps, with x_0 given,

$$\text{(i)} \quad x_{n+1} = 2x_n - 3n + 4,$$

$$\text{(ii)} \quad x_{n+1} = \frac{1}{2} x_n + 2^n,$$

$$\text{(iii)} \quad x_{n+1} = 4x_n + 3 \times 4^n, \qquad x_0 = 1.$$

4. (a) Find the fixed points and their (linear) stability of the iterated map

$$x_{n+1} = x_n^2 - 2x_n + 2.$$

(b) Show that the nonlinear map

$$x_{n+1} = x_n^2 - \mu x_n, \qquad (\text{with } \mu \geq 0)$$

has the fixed points $x_s^{(1)} = 0$ and $x_s^{(2)} = \mu + 1$.
Show that $x_s^{(2)} = \mu + 1$ is unstable for all $\mu \geq 0$ and that $x_s^{(1)} = 0$ is stable provided $\mu < 1$.

Show that this map has period 2 points $x_{1,2}$ given by

$$x_{1,2} = \frac{(\mu - 1) \pm \sqrt{(\mu - 1)(\mu + 3)}}{2}, \qquad \text{for } \mu > 1.$$

Show that these period 2 points are stable for $1 < \mu < \sqrt{6} - 1$.

5. (a) Show that the eigenvalues of the matrix

$$\begin{pmatrix} 4 & -1 \\ -4 & 4 \end{pmatrix}$$

are $\lambda_1 = 2$, $\lambda_2 = 6$ with corresponding eigenvectors $\mathbf{e}_1 = (1, 2)^T$ and $\mathbf{e}_2 = (1, -2)^T$.

(b) Use this result to find the solution to the two-dimensional linear map

$$\begin{aligned} x_{n+1} &= 4x_n - y_n \\ y_{n+1} &= -4x_n + 4y_n \end{aligned}$$

with $x_0 = 1$, $y_0 = 1$.

END