MATH161001

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH1610**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH1610
(May/June 2006)

Discrete Systems

Time allowed: **2 hours**

Do not attempt more than **4** questions. All questions carry equal weight.

1. (a) If $\mathbf{a} = (3, 1, -2)$ and $\mathbf{b} = (-2, 3, 4)$, find
(i) $\mathbf{a} + 3\mathbf{b}$,
(ii) $|2\mathbf{a} - \mathbf{b}|$,
(iii) a unit vector in the direction of $\mathbf{a} - 2\mathbf{b}$,
(iv) the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$,
(v) $\mathbf{a} \times \mathbf{b}$.

(b) Find the equation of the line L_1 through the point with position vector $(1, 2, -1)$ and parallel to the vector $(1, 4, -2)$.
Find the equation of the line L_2 through the points with position vectors $(5, 1, -4)$ and $(8, -4, -5)$.
Show that the lines L_1 and L_2 intersect and find their point of intersection.

2. (a) Find a unit vector perpendicular to the vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.

(b) (i) Find the equation of the plane Π_1 through the point with position vector $(1, 4, 6)$ and perpendicular to the vector $(2, 1, 4)$.
Determine the perpendicular distance of the plane Π_1 from the origin.
(ii) Find the equation of the plane Π_2 which passes through the three points $(1, 2, -3)$, $(3, -2, 5)$ and $(4, -1, 2)$.
(iii) Find the angle between the planes Π_1 and Π_2.

3. (a) Newton's method gives rise to the iterated map

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{p}{x_n}\right), \qquad x_0 \neq 0 \text{ given}$$

for finding $\sqrt{p}$. Use this map to find $\sqrt{5}$, starting with $x_0 = 1.0$. You should show your working and give your result correct to 5 decimal places.

(b) For the one-dimensional iterated map $x_{n+1} = f(x_n)$,
(i) say what is meant by a fixed point of the map and give the condition for the (linear) stability of this fixed point,
(ii) say what is meant by a period 2 point of this map.
(iii) If x_1 and x_2 are period 2 points of the map, establish the result that

$$\left(\frac{df^2(x)}{dx}\right)_{x_1} = f'(x_1)f'(x_2),$$

and indicate how this result may be used to determine the stability of a period 2 point.

(c) Find the fixed points and their (linear) stability of the iterated map

$$x_{n+1} = x_n^2 - 6x_n + 12.$$

4. (a) Find the fixed points of the map

$$x_{n+1} = \mu + x_n - x_n^2, \qquad (\text{with } \mu > 0)$$

and determine their stability.
Show that this map has the period 2 points $x_{1,2}$ and these are given by

$$x_{1,2} = 1 \pm \sqrt{\mu - 1}, \qquad \mu > 1.$$

Show that these period 2 points are stable for $1 < \mu < \frac{3}{2}$.

(b) Find the general solution to the non-homogeneous linear maps, with x_0 given,

$$\text{(i)} \quad x_{n+1} = 4x_n + 6n - 5,$$

$$\text{(ii)} \quad x_{n+1} = 3x_n + 2 \times 4^n,$$

$$\text{(iii)} \quad x_{n+1} = 5x_n + 5^n, \qquad x_0 = 1.$$

5. (a) Find the solution to the two-dimensional map

$$\begin{aligned} x_{n+1} &= x_n - y_n \\ y_{n+1} &= 2x_n + 4y_n \end{aligned}$$

with $x_0 = 1$, $y_0 = 1$.

(b) Indicate how the second-order map

$$x_{n+1} = 4x_n - 3x_{n-1}$$

can be expressed as a two-dimensional map. By considering the eigenvalues of the associated matrix, show that $x_n \propto 3^n$ for n large.

END