MATH–133101

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–1331

Only approved basic scientific calculators may be used.

Examination for the Module MATH–1331

(May/June 2001)

Linear algebra with applications

Time allowed : 3 hours

Answer **four** questions.
All questions carry equal marks.

1. **(a)** A student is taking an exam consisting of 50 multichoice questions and 10 longer problems. She has 90 minutes to take the exam and knows that she cannot possibly answer every question. Multichoice questions are each worth 5 points and take 2 minutes. Problems are each worth 20 points and take 10 minutes. The student is required to answer at least 10 multichoice questions and at least 3 problems. Let x be the number of problems and y the number of multichoice questions to be answered. Write down the inequalities to be satisfied by x and y, draw the feasible set and find how many of each type of question the student should answer.

(b) Use the Gaussian reduction process to find the complete solution set of the system of equations

$$\begin{aligned} 2w - x + y - 2z &= 0 \\ v - w - 2x + z &= 0 \\ 2v - 5x + y - 3z &= 0 \\ v - 5w - 2y + 5z &= 0. \end{aligned}$$

Find a basis for the solution space. What is the dimension of the solution space?

(c) Find a basis for each of the following subspaces of $\mathbb{R}^4$:
(*i*) $\{(w, x, y, z) : w + 2x = 0 \text{ and } 3y + 4z = 0\}$,
(*ii*) $\{(a + b, a - b, 2a + b, 2a - b)) : a, b \in \mathbb{R}\}$.

2. **(a)** For each of the following subsets S of $\mathbb{R}^3$, determine whether S is a spanning set, and whether it is linearly independent:
(*i*) $S = \{(1, 5, -3),\ (3, 1, 7),\ (2, 3, 2)\}$,
(*ii*) $S = \{(1, 5, -3),\ (3, 1, 7)\}$,
(*iii*) $S = \{(1, 5, -3),\ (3, 1, 7),\ (2, -1, 4)\}$,
(*iv*) $S = \{(1, 5, -3),\ (3, 1, 7),\ (2, 3, 2),\ (2, -1, 4)\}$.

Question 2 continues . . .

(b) Solve the equation $\begin{vmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 2 & 5 \\ 3 & 2 & 5 & -1 \\ 2 & 5 & 2 & x \end{vmatrix} = 0.$

(c) Find AB and BA, where $A = \begin{bmatrix} 2 & -3 & 4 & -5 \\ 1 & 0 & 7 & 3 \end{bmatrix}$, $\quad B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 5 & 0 \\ 2 & 5 \end{bmatrix}$.

(d) Find an elementary matrix E such that $E \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$.

3. **(a)** Calculate the inverse of the matrix $\begin{bmatrix} 1 & 3 & 2 & 5 \\ 2 & 7 & 4 & 10 \\ -1 & -2 & -1 & 3 \\ 1 & 2 & 1 & -2 \end{bmatrix}$.

Use your answer to solve the equations

$$\begin{aligned} w + 3x + 2y + \ \ 5z &= 1 \\ 2w + 7x + 4y + 10z &= 1 \\ -w - 2x - \ \ y + \ \ 3z &= 1 \\ w + 2x + \ \ y - \ \ 2z &= 1. \end{aligned}$$

(b) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 4 & 1 & 1 \\ -3 & 0 & -3 \\ -2 & -2 & 1 \end{bmatrix}$.

Write down a matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

(c) A corporation has a plastics division and a machinery division. For each £1 of output, the plastics division requires 6p worth of plastics and 12p worth of machinery. For each £1 of output, the machinery division needs 7p worth of plastics and 14p worth of machinery. How much should each division produce to meet a demand for £880,000 worth of plastics and £720,000 of machinery?

4. **(a)** Use the simplex method to maximise $2x + 3y$ subject to the constraints

$$x + 2y \leqslant 14, \quad x + y \leqslant 9, \quad 3x + 2y \leqslant 24, \quad x \geqslant 0, \quad y \geqslant 0.$$

(b) Explain how to choose the pivot element for the simplex tableau

$$\begin{bmatrix} 4 & 3 & 15 & 1 & 0 & 0 & 0 & 30 \\ 2 & 5 & 10 & 0 & 1 & 0 & 0 & 25 \\ 1 & -2 & 3 & 0 & 0 & 1 & 0 & -15 \\ -1 & -2 & -12 & 0 & 0 & 0 & 1 & -75 \end{bmatrix}.$$

Indicate the pivot element clearly, but do not carry out the pivoting operation.

Question 4 continues . . .

(c) Explain why the stochastic matrix $A = \begin{bmatrix} 2/3 & 1/3 & 1/3 \\ 0 & 0 & 2/3 \\ 1/3 & 2/3 & 0 \end{bmatrix}$ is regular. What is $\lim_{n\to\infty} A^n$?

5. (a) A Markov process has the transition matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 2/5 & 0 \\ 0 & 0 & 1/5 & 1/6 \\ 0 & 0 & 2/5 & 1/2 \end{bmatrix}.$$

What is the stable matrix of this process?

If initially all four states are equally distributed, what is the stable state distribution?

(b) A smuggler attempts to bring illegal consignments of cigarettes into an island having two ports. Each day the coast guard is able to patrol only one of the ports. If the smuggler enters via an unpatrolled port, he will be able to sell his cigarettes for £7,000. If he enters the first port and it is patrolled that day, he is certain to be caught and will have his cigarettes (worth £1,000) confiscated and be fined £1,000. If he enters the second port (which is big and crowded) and it is patrolled that day, he will have time to jettison his cargo and thereby escape a fine. Find the optimal strategy for the smuggler.

How profitable is cigarette smuggling (that is, what is the expected value of this game)?

END