MATH–115001

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–115001

Only approved basic scientific calculators may be used.

Examination for the Module MATH–1150

(May/June 2007)

Mathematics for Geophysics 2

Time allowed: 2 hours

Answer **four** questions.
All questions carry equal marks.

1. **(a)** In each of the following cases, find the function $f(t)$ for which its Laplace transform, $\bar{f}(p)$ is given by

(i) $\dfrac{1}{p+2}$ (ii) $\dfrac{1}{(p+2)(p+3)}$

(iii) $\dfrac{p}{(p+2)(p+3)}$ (iv) $\dfrac{p}{(p+2)^2}$

(v) $\dfrac{p^2}{(p+2)(p+3)}$

(b) A seismometer has a transfer function with two zeros at 0 and two poles at -2 and -3. Write down the transfer function and, using your results from part (a) or otherwise, compute the response of the seismometer to a unit impulse at $t = 0$.

You may use without proof any standard Laplace transform results including the shift theorem.

2. **(a)** For each of the following second order ordinary differential equations, find their general solution.

(i) $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = \sin x$$

(ii) $$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 5y = e^x$$

(b) Solve the following second order ordinary differential equation with the given boundary conditions:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2x + 5\ , \qquad y(0) = 9,\ y(1) = 11 + e.$$

3. Solve the following first order ordinary differential equations with the given boundary condition:

(a) $\dfrac{dy}{dx} = y^2 \cos x$ $\qquad y(0) = 1$

(b) $\dfrac{dy}{dx} - 3\dfrac{y}{x} = \dfrac{2}{x^2}$ $\qquad y(1) = 2$

(c) $\dfrac{dy}{dx} = \dfrac{y}{x} + e^{(y/x)}$ $\qquad y(1) = 0$

(d) $-\dfrac{dy}{dx} \sin y \sin x + \cos y \cos x = 1$ $\qquad y(\frac{\pi}{2}) = 0$

4. **(a)** Let $f(x, y) = \sin(x + e^y)$. Find the Taylor series to second order about the point $(\pi - 2, \ln 2)$.

(b) The function $u(x, y)$ is expressible in terms of the polar coordinates (r, θ) through the relations

$$x = r\cos\theta, \qquad y = r\sin\theta$$

(i) By solving for r and θ in terms of x and y, find the partial derivatives

$$\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}$$

(ii) Hence using the chain rule, show that

$$\frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}, \qquad \frac{\partial u}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

(iii) Using the chain rule again, find the derivatives

$$\frac{\partial u^2}{\partial x^2}, \frac{\partial u^2}{\partial y^2}$$

in terms of r, θ and derivatives of u with respect to r and θ, and show that

$$\frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2}$$

continued . . .

5. **(a)** For each of the following functions

$$\begin{array}{ll} \text{(i)} & f(x) = \cos x \\ \text{(ii)} & f(x) = |\sin(2x)| \\ \text{(iii)} & f(x) = \sinh x \end{array}$$

(1) sketch the graphs in the range $-2\pi \leq x \leq 2\pi$,

(2) determine which are periodic, for those that are, state the period and

(3) state whether the function is even, odd or neither.

(b) Let f be defined by $f(x) = x$ over the interval $[0, \pi]$.

(i) If $f(x)$ is **even** over the interval $[-\pi, \pi]$ and periodic with period 2π, find its Fourier series.

(ii) If $f(x)$ is **odd** over the interval $[-\pi, \pi]$ and periodic with period 2π, find its Fourier series.

6. A second order ordinary differential equation is defined as

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 9,$$

with the boundary conditions $y(0) = y'(0) = 1$.

(a) By trying solutions of the form $y = e^{\lambda t}$, solve the differential equation using the standard method.

(b) By taking the Laplace transform of the differential equation, show that $\bar{y}(p) = \mathcal{L}\big(y(t)\big)$ can be written

$$\bar{y}(p) = \frac{1}{p} + \frac{1}{p^2 - 6p + 9}.$$

Hence invert the Laplace transform to find the solution of the differential equation.

You may use without proof any standard Laplace transform results including the shift theorem.

END