MATH–115001

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–115001

Only approved basic scientific calculators may be used.

Examination for the Module MATH–1150

(May/June 2005)

Mathematics for Geophysics 2

Time allowed: 2 hours

Answer **four** questions.
All questions carry equal marks.

1. **(a)** Find the Taylor series about the point $(0, 1)$ of the function

$$f(x, y) = (e^x + y^2 - 1)^{10}$$

up to and including the quadratic terms.

(b) Find the critical points of the function

$$f(x, y) = x^4 - 2x^2 + y^3 - 3y + 1$$

and determine whether they are minima, maxima or saddle points.

2. Solve the following first order ordinary differential equations with the given boundary condition:

(a) $$\frac{dy}{dx} = \frac{y^3}{x^4} \qquad y(1) = 1$$

(b) $$\frac{dy}{dx} + \frac{y}{x} = x^3 \qquad y(1) = 2$$

(c) $$\frac{dy}{dx} - \frac{2xy}{1 + x^2} = 1 \qquad y(0) = 0$$

(d) $$\frac{dy}{dx} \sin x + y \cos x = -1 \qquad y(\frac{\pi}{2}) = 0$$

3. **(a)** Find the general solution to the following second order ordinary differential equation:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = x$$

(b) Solve the following second order ordinary differential equations with the given boundary conditions:

(i) $$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0 \qquad y(0) = 0, y'(0) = 1$$

(ii) $$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{2x} \qquad y(0) = 0, y'(0) = 0$$

4. **(a)** Let $f(x, y, z) = \cos(x^2y + \ln z)$. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ and $\frac{\partial^2 f}{\partial x \partial z}$.

(b) A function $f(x, y)$ is defined in terms of variables x and y, which in turn may be expressed as

$$x = e^s \cos\theta \qquad\qquad y = e^s \sin\theta$$

By first solving for s and θ in terms of x and y, compute the derivatives

$$\frac{\partial s}{\partial x}, \ \frac{\partial s}{\partial y}, \ \frac{\partial \theta}{\partial x} \text{ and } \frac{\partial \theta}{\partial y}$$

Hence express the derivatives

$$\frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y} \text{ and } \frac{\partial f}{\partial x^2}$$

in terms of s and θ, and derivatives of f with respect to s and θ.

5. **(a)** Sketch the graph of the periodic extension of period 2π of the function

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & \pi \le x \le 2\pi \end{cases}$$

and find its **complex** Fourier series.

(b) A function $f(x)$ is defined over the half interval $[0, \pi]$ as

$$f(x) = x + 2$$

If $f(x)$ is even, sketch the periodic extension of period 2π of f and find its Fourier series.

You may use the fact that, if n is a non-zero integer then

$$\int_0^\pi x \cos nx \, dx = \frac{1}{n^2}(\cos n\pi - 1)$$

continued ...

6. **(a)** The response $y(t)$ at time t of a linear system with input $u(t)$ satisfies the equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 2\frac{du}{dt} + 6u$$

Assuming that all initial conditions are zero (i.e. y, $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$, u and $\frac{du}{dt}$ are all zero for $t < 0$), show that the Laplace transform $\bar{y}(p)$ of y and $\bar{u}(p)$ of u are related by

$$\bar{y}(p) = \frac{2p+6}{p^2+2p+5}\bar{u}(p)$$

[You may quote standard results regarding the Laplace transform of the derivatives of a function. If required in this or any other part of question 6, you may also quote and use the shift theorem and the fact that the Laplace transforms of e^{at}, $\sin(at)$ and $\cos(at)$ where a is a real number are given by $1/(p-a)$, $a/(p^2+a^2)$ and $p/(p^2+a^2)$ respectively.]

Hence, find the response $y(t)$ for $t > 0$ to the following inputs $u(t)$:

(i) $\quad u(t) = e^{-3t}$

(ii) $\quad u(t) = \delta(t)$

(b) The response of a simple seismometer is determined by its transfer function which has two zeros at 0 and poles at -1 and -3. Write down the transfer function (up to a constant multiple) and hence determine its response to a unit delta function impulse at $t = 0$, assuming that all initial conditions are zero.

END