MATH–106001

This question paper consists of 4 printed pages, each of which is identified by the reference MATH–106001

Only approved basic scientific calculators may be used.

Examination for the Module MATH–1060

(May/June 2004)

Intro Linear Algebra

Time allowed: 2 hours

Answer **four** questions.
All questions carry equal marks.

1. (a) For a system of equations:

$$\begin{aligned} -x+3y+2z &= -8 \\ x+z &= 2 \\ 3x+3y+az &= b. \end{aligned}$$

Find the values of a and b for which the system **(i)** has no solutions; **(ii)** has a unique solution; **(iii)** has infinitely many solutions.

(b) (i) Evaluate the determinant

$$\begin{vmatrix} -1 & -1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & -1 & 2 \end{vmatrix};$$

(ii) Find all real numbers x and y such that the determinant

$$\begin{vmatrix} 0 & x & y \\ y & 0 & x \\ x & y & 0 \end{vmatrix} \quad \text{is equal to } 0;$$

2. **(a)** Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 5 & 4 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

(i) State which of the following exists, evaluating those which do: AB, BA, AD, CA, DCA, $AB + CA$.

(ii) Show that $CD + DC \neq I_{3\times3}$, where $I_{3\times3}$ is the 3×3 identity matrix.

(b) Let A be a 2×2 matrix and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. If A and B commute (i.e. $AB = BA$) show that $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ for some numbers a and b.

(c) Let $H = \begin{pmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$.

(i) By using elementary row operations, find the inverse of H, (*show all steps in your working. No marks will be given for just writing down the answer.*)

(ii) Solve the system of equations $H\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$.

(iii) Solve the system of equations $H\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

3. **(a)** Let V be the set of all pairs (x, y) where x, y are real numbers. Define an addition of pairs by the standard rule: $(x, y) \oplus (u, v) = (x + u, y + v)$ and a new "scalar multiplication" $\circ$, by: $\lambda \circ (x, y) = (\lambda y, \lambda x)$ (*note the order*!), λ being any real number. Recalling that for a vector space the second multiplication axiom involves the equality

$$\lambda \circ (a + b) = \lambda \circ a + \lambda \circ b$$

and that the fourth multiplication axiom involves the equality

$$(\lambda\mu) \circ a = \lambda \circ (\mu \circ a) = \mu \circ (\lambda \circ a).$$

(i) Show that the second axiom holds for all a, $b \in V$ and $\lambda \in \mathbb{R}$;

(ii) $W = \{(x, y, z) : x - 2y - z = 0\}$;

(iii) $W = \{(x, y, z) : x, y, z \text{ are all even numbers } \}$;

(iv) $W = \{(x, y, z) : (x + y)z = 0\}$;

(v) $W = \{(2u - v, 2v - w, 2w - u) : u, v, w \in \mathbb{R}\}$;

(c) Let A, B be subspaces of the vector space V. Prove that the intersection $A \cap B$ of subspaces A and B is a subspace of V as well.

4. (a) Let V be a vector space. Explain what it means to say that

(i) the set $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ of p vectors in V *spans* V;

(ii) the set $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_q$ of q vectors in V are *linearly independent*;

(iii) the set $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ of r vectors in V is a *basis* of V;

(iv) given that (i), (ii), and (iii) above are true statements in V, state any relationship you know of between the integers p, q, and r.

(b) Let $A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & -1 & -1 & 3 \\ 3 & 0 & -2 & 5 \\ -5 & 1 & 3 & -8 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$, and $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

(i) Find a basis for the row space of A;

(ii) Find a basis for the solution space of $A\mathbf{x} = \mathbf{0}$.

(iii) Using (ii) find two different solutions $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$, of $A\mathbf{x} = \mathbf{0}$ for which $y = 1$.

(b) Find the eigenvalues and corresponding eigenvectors for the matrix $B = \begin{pmatrix} 8 & -18 \\ 3 & -7 \end{pmatrix}$.
Hence find the element $c_{2,1}$ in the $(2,1)$ place of the 2×2 matrix $C = B^5$.

(c) $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are functions of t which are solutions of the system of differential equations

$$\begin{aligned} \dot{\mathbf{x}}_1 &= 8\mathbf{x}_1 - 18\mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= 3\mathbf{x}_1 - 7\mathbf{x}_2 \end{aligned}$$

Express $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ in terms of the exponential function, given that $\mathbf{x}_1(0) = 2$ and $\mathbf{x}_2(0) = 3$.

END