MATH–106001

Only approved basic scientific calculators may be used.

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–1060

Examination for the Module MATH–1060
(June 2006)

Introductory Linear Algebra

Time allowed : 2 hours

Do not answer more than four questions. All questions carry equal marks.

1. **(a)** For a system of equations:

$$\begin{aligned} x - 2y + 3z &= 1 \\ 2x + ky + 6z &= 6 \\ -x + 3y + (k-3)z &= 0. \end{aligned}$$

Find the values of k for which the system (i) has no solutions; (ii) has a unique solution; (iii) has infinitely many solutions. In the case (iii) write down the general solution to the system.

(b) (i) Evaluate the determinant

$$\begin{vmatrix} 1 & 3 & 2 \\ 8 & 4 & 0 \\ 2 & 1 & 2 \end{vmatrix}.$$

(ii) Find a value of x for which the determinant

$$\begin{vmatrix} 3 & 3+x & 1 \\ 1 & 2 & 3 \\ x+2 & 1 & 2+x \end{vmatrix} \quad \text{is equal to 0.}$$

(In all cases show your working. Merely writing down the answers will gain you no marks.)

(c) Give a specific example of a 2×2 matrix A such that $\det(2A) = 2 \det A$.

2. **(a)** Let $A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$, $D = \begin{pmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{pmatrix}$.

(i) State which of the following exists, evaluating those which do: AB, BA, AC, CA, $AB - CA$.

(ii) Show that $CD \neq DC$.

Question 2 continues . . .

(b) Given that x and y are real numbers with $x > y$, find the unique matrix A of the form $\begin{pmatrix} x & 1 \\ 2 & y \end{pmatrix}$ which satisfies the equation

$$A^2 + 5A = -4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) Let $H = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 3 & 4 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 9 \\ -11 \\ 13 \end{pmatrix}$.

(i) By using elementary row operations, find the inverse of H, (show all steps in your working. No marks will be given for just writing down the answer.)

(ii) Solve the system of equations $H\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$.

(iii) Solve the system of equations $H\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

3. **(a)** Let V be the set of all pairs (x, y) where x, y are real numbers. Define a new operation $\oplus$ of "addition of pairs" by: $(x, y) \oplus (u, v) = (x+u, y+v)$ and a new "scalar multiplication" $\circ$, by: $r \circ (x, y) = (-x, y)$, for any real number r. Recalling that the vector space second multiplication axiom involves the equality $\lambda \circ (a \oplus b) = \lambda \circ a \oplus \lambda \circ b$ and that the third multiplication axiom involves the equality $(\lambda + \mu) \circ a = \lambda \circ a \oplus \mu \circ a$, show that the second multiplication axiom holds for all a, $b \in V$ and $r \in \mathbb{R}$ and show, by means of example, that the third multiplication axiom may fail in V.

(b) Let V be a vector space. Explain what is meant by a *subspace* of V.

For each of the following subsets W of $\mathbb{R}^3$ determine whether or not W is a subspace of $\mathbb{R}^3$. (Here addition and scalar multiplication are as usual for $\mathbb{R}$. If the subset is *not* a subspace give a specific example to indicate why it is not a subspace.

(i) $W = \{(x, y, z) : -x + y + 2z = 2\}$;

(ii) $W = \{(x, y, z) : x + 2y - z = 0\}$;

(iii) $W = \{(x, y, z) : x + yz = 0\}$;

(iv) $W = \{(x, y, z) : x - z \text{ is an integer }\}$;

(v) $W = \{(u - v, v - w, w - v) : u, v, w \in \mathbb{R}\}$;

(c) Let A, B be subspaces of the vector space V. Define $A - B$ to be the subset of V comprising all elements of the form $a - b$ where $a \in A$ and $b \in B$. Prove that $A - B$ is a subspace of V.

4. **(a)** Let V be a vector space. Explain what it means to say that

Question 4 continues . . .

(i) the set $\mathbf{u}_1$, $\mathbf{u}_2$, $\ldots$, $\mathbf{u}_r$, of vectors in V are *linearly independent*;

(ii) the set $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_s$, of vectors in V *spans* V;

(iii) the set $\mathbf{w}_1$, $\mathbf{w}_2$, $\ldots$, $\mathbf{w}_t$, of vectors in V is a *basis* of V;

(iv) given that (i), (ii), and (iii) above are true statements in V, state any relationship you know of between the integers r, s, and t.

(b) Let $A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 2 & 3 & -2 & 3 \\ 3 & 4 & -5 & 7 \\ 1 & 1 & -3 & 4 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$, and $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

(i) Find a basis for the row space of A;

(ii) Find a basis for the solution space of $A\mathbf{x} = \mathbf{0}$.

(iii) Using (ii) find two solutions $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$, of $A\mathbf{x} = \mathbf{0}$ for which $y = -1$.

(c) Show that $(3, 5, -1, 4)$ is not in the row space of A.

5. **(a)** Let λ be an eigenvalue for $n \times n$ matrix A and let $\mathbf{x}$ be a corresponding eigenvector. Let $I_{n\times n}$ be the identity $n \times n$ matrix.

(i) Show that $\mathbf{x}$ is an eigenvector of the matrix $A^2 + rA$ for any real number r. Find the corresponding eigenvalue.

(ii) If $A^3 = I_{n\times n}$ what are the possible eigenvalues of A?

(b) Find the eigenvalues and corresponding eigenvectors for the matrix $B = \begin{pmatrix} -3 & 15 \\ -2 & 8 \end{pmatrix}$. Hence find the element in the $(1, 2)$ place of the 2×2 matrix B^4.

(c) $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are functions of t which are solutions of the system of differential equations

$$\begin{aligned} \dot{\mathbf{x}}_1 &= -3\mathbf{x}_1 + 15\mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= -2\mathbf{x}_1 + 8\mathbf{x}_2 \end{aligned}$$

Express $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ in terms of the exponential function, given that $\mathbf{x}_1(0) = 1$ and $\mathbf{x}_2(0) = 0$.

END