King’s College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the Authority of the Academic Board.

B.Sc. EXAMINATION

CP2720 Numerical Problems in Physics

Summer 2005

Time allowed: THREE Hours

Candidates should answer all SIX questions from SECTION A,
and no more than TWO questions from SECTION B.
No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Physical Constants

Permittivity of free space	ε_0	=	8.854×10^{-12}	$\mathrm{F\ m^{-1}}$
Permeability of free space	μ_0	=	$4\pi \times 10^{-7}$	$\mathrm{H\ m^{-1}}$
Speed of light in free space	c	=	2.998×10^{8}	$\mathrm{m\ s^{-1}}$
Gravitational constant	G	=	6.673×10^{-11}	$\mathrm{N\ m^{2}\ kg^{-2}}$
Elementary charge	e	=	1.602×10^{-19}	C
Electron rest mass	m_e	=	9.109×10^{-31}	kg
Unified atomic mass unit	m_u	=	1.661×10^{-27}	kg
Proton rest mass	m_p	=	1.673×10^{-27}	kg
Neutron rest mass	m_n	=	1.675×10^{-27}	kg
Planck constant	h	=	6.626×10^{-34}	J s
Boltzmann constant	k_B	=	1.381×10^{-23}	$\mathrm{J\ K^{-1}}$
Stefan-Boltzmann constant	σ	=	5.670×10^{-8}	$\mathrm{W\ m^{2}\ K^{-4}}$
Gas constant	R	=	8.314	$\mathrm{J\ mol^{-1}\ K^{-1}}$
Avogadro constant	N_A	=	6.022×10^{23}	$\mathrm{mol^{-1}}$
Molar volume of ideal gas at STP		=	2.241×10^{-2}	$\mathrm{m^{3}}$
One standard atmosphere	P_0	=	1.013×10^{5}	$\mathrm{N\ m^{-2}}$

SECTION A – Answer all SIX parts of this section

In questions asking for descriptions of methods of solution, you are not expected to work out accurate answers or write a computer program.

1.1) What errors might be introduced into the calculation of a function if a series that converges very slowly is used?

[7 marks]

1.2) Define, and describe how to evaluate, the continued fraction

$$\tan x = \frac{x}{1-}\,\frac{x^2}{3-}\,\frac{x^2}{5-}\,\frac{x^2}{7-}\,\frac{x^2}{9-}\cdots.$$

for a given value of x, ensuring accuracy to a given level.

[7 marks]

1.3) The sine integral is given by

$$Si(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}t$$

Describe how it could be evaluated numerically for a given value of x.

[7 marks]

1.4) Use a recurrence relation stability test to show that the following recurrence relation is stable for all values of n and θ.

$$\cos(n\theta) = 2\cos\theta\cos[(n-1)\theta] - \cos[(n-2)\theta]$$

[7 marks]

1.5) A partial differential equation for a quantity $f(x,t)$ is

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x}.$$

Express this equation in finite difference form using the Forward Time Centred Space (FTCS) scheme and show that the finite difference equation is unstable in this form.

[7 marks]

1.6) For what type of problem is simulated annealing a suitable minimisation technique? Justify your answer.

[7 marks]

SECTION B - Answer TWO questions

Descriptions of methods of numerical or computational solution are required, not computer programs or actual numerical answers.

2) The distance, s, an aircraft moves along a runway, under a constant thrust from its engines is given by

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} - B\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = C$$

where B and C are constants depending on the properties of the aircraft (including its weight) and the magnitude of the thrust. Write this second order differential equation in the form of two interdependent first order differential equations. Assuming that the aircraft starts from rest at the start of the runway, write down the boundary conditions.

[6 marks]

Describe how the 4th order Runge Kutta method could be used to solve this equation numerically, if the values of the constants are known.

[20 marks]

In order to take off, the aircraft must reach a given velocity, V, by the time it reaches the end of the runway (s = S). Describe how to use the method to determine the maximum weight of the aircraft as a function of runway length.

[4 marks]

3) A sound wave in a pipe is transmitted through the air according to the wave equation in one dimension,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

where $y(x,t)$ is the displacement of the air at a distance x down the pipe, and v is the velocity of sound. Write down the finite difference (FD) version of this equation.

[5 marks]

At $t = 0$, there is no sound in the pipe. Thereafter, at one end of the pipe, an oscillation, $y(0,t) = a \sin \omega_0 t$, is generated. The other end is closed, such that $y(X,t) = 0$. How could these boundary conditions be imposed on the finite difference equations, given the values of the constants v, a, X and ω_0?

[10 marks]

Describe how to build up a numerical solution of the finite difference equations giving the displacement along the pipe as a function of time.

[5 marks]

Discuss how to select the time and distance step in the FD scheme.

[10 marks]

4) The gamma function $\Gamma(a)$ arises in quantum mechanics problems involving Coulomb potentials. It has the properties that:

i) for all values of a $\Gamma(a+1) = a\Gamma(a)$;

ii) when a is an integer $\Gamma(a) = (a-1)!$

iii) $\Gamma(0.5) = \sqrt{\pi}$.

List values of $\Gamma(a)$, for a = 1, 1.5, 2, 2.5, 3, 3.5.

[5 marks]

There are three possible ways of calculating the incomplete gamma function, $P(x,a)$.

a) The integral definition is

$$P(x,a) = \frac{1}{\Gamma(a)}\int_0^x e^{-t}t^{a-1}\mathrm{d}t \qquad \text{for } a > 0.$$

b) The series expansion is given by

$$P(x,a) = e^{-x}x^a\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(a+1+n)},$$

which is convergent for all $x > 0$ and $a > 0$, but which converges rather slowly when $x > a+1$.

c) The continued fraction expansion is

$$P(x,a) = \frac{e^{-x}a^x}{\Gamma(a)}\left(\frac{1}{x+}\,\frac{1-a}{1+}\,\frac{1}{x+}\,\frac{2-a}{1+}\,\frac{2}{x+}\,\frac{3-a}{1+}\cdots\right),$$

which is also convergent for all $x > 0$ and $a > 0$, but which converges slowly for $x < a+1$.

Explain how to evaluate $P(x,a)$ in the range $0 < x \le 4$, from the information given above, for $a = 1.0$ and $a = 2.5$, in the most efficient way (i.e. in the way which requires the least number of calculation steps for a given accuracy).

[25 marks]