King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2201 INTRODUCTORY QUANTUM MECHANICS

Summer 1997

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A,
and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Pauli matrices are given by

$$\boldsymbol{S}_x = \frac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{S}_y = \frac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{S}_z = \frac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

SECTION A – Answer SIX parts of this section

1.1) A particle moving in one dimension is confined to the interval $| x | \leq a$ and is described by the wave function

$$\psi(x) = A \cos \frac{\pi x}{2a},$$

where A is a normalization constant. Determine A and find the probability that the particle is in the region $x \geq 0$.

Note: $\cos 2\theta = 2\cos^2\theta - 1$

[7 marks]

1.2) Define an hermitian operator and prove that all its eigenvalues are real.

[7 marks]

1.3) At a given instant, a quantum harmonic oscillator is in a state described by the normalized wave function

$$\psi(x) = \sqrt{\frac{1}{3}} u_0(x) + \sqrt{\frac{1}{6}} u_1(x) + \sqrt{\frac{1}{2}} u_4(x) \ ,$$

where $u_n(x)$ is the normalized energy eigenfunction of the oscillator corresponding to an eigenvalue $E_n = (n + \frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \ldots$. What are the possible results of a measurement of the energy of this system and what are their relative probabilities? Using these probabilities, show that the expectation value of the energy of the oscillator is $\frac{8}{3}\hbar\omega$.

[7 marks]

1.4) Give the Schrödinger representation of the operators $\boldsymbol{z}$ and $\boldsymbol{p}_z$ representing the position z and the $z-$component of the linear momentum, respectively. By operating on a general wave function $\psi(x, y, z)$, prove that

$$[\boldsymbol{z}, \boldsymbol{p}_z] \equiv \boldsymbol{z}\boldsymbol{p}_z - \boldsymbol{p}_z\boldsymbol{z} = i\hbar.$$

Are the operators $\boldsymbol{z}$ and $\boldsymbol{p}_z$ *compatible* or *incompatible*? What does this imply physically?

[7 marks]

1.5) Explain briefly what is meant by the *correspondence principle.* Use it to derive representation-free operators for the cartesian components of the angular momentum operator $\boldsymbol{L}$.

[7 marks]

1.6) Write down Schrödinger's equation of motion for a particle confined to the $x-$axis. Derive particular solutions in which the spatial (x) and temporal (t) variables are separated.

[7 marks]

1.7) An electron is in the spin state

$$\psi = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ i \end{pmatrix} .$$

Calculate the expectation value in this state of the spin component $\boldsymbol{S}_z$. What is the probability that, on measurement, the electron will be found in the spin-up state?

[7 marks]

1.8) Write down the Schrödinger equation for the hydrogenic atom. Discuss the r, θ and ϕ-dependence of the bound state solutions paying particular attention to the role played by the quantum numbers n, ℓ and m_ℓ.

[7 marks]

SECTION B – Answer TWO questions

2) Describe **two** of the following:

(i) the photo-electric effect and the way in which it provides evidence for the particle aspects of electromagnetic radiation,

[15 marks]

(ii) the Davisson and Germer experiments and the way in which they provide evidence for the existence of matter waves,

[15 marks]

(iii) the Bohr model of the hydrogen atom and the way in which it provides a description of the optical spectrum of hydrogen,

[15 marks]

(iv) the Stern-Gerlach experiment and the way in which it provides evidence for the existence of electron spin.

[15 marks]

3) A beam of neutral spin-$\frac{1}{2}$ particles travelling in the $y-$direction passes through a Stern-Gerlach apparatus. The spins in the two exit beams are in the eigenstates of $\boldsymbol{S}_x$,

$$\alpha_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \text{and} \qquad \beta_x = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}.$$

All particles in the entry beam are in the eigenstate α_θ corresponding to the eigenvalue $+\frac{1}{2}\hbar$ for the component of spin angular momentum in a direction aligned at an angle θ to the positive $z-$axis in the $xz-$plane. Prove that

$$\alpha_\theta = \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}$$

[15 marks]

and show that the relative intensities of the exit beams are $\frac{1}{2}(1 \pm \sin\theta)$.

[15 marks]

4) In spherical polar coordinates (r, θ, ϕ), the components of the orbital angular momentum operator are

$$\begin{aligned}
\boldsymbol{L}_x &= i\hbar(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}),\\
\boldsymbol{L}_y &= i\hbar(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}),\\
\boldsymbol{L}_z &= -i\hbar\frac{\partial}{\partial\phi}.
\end{aligned}$$

State the commutation relations between pairs of angular momentum components and compare your results with the corresponding relations between components of the linear momentum. What are the physical implications of these commutation relations?

[7 marks]

Show that the function

$$Y_{1,0} = \cos\theta$$

is an eigenfunction of the operator $\boldsymbol{L}_z$ and determine the corresponding eigenvalue.

[3 marks]

Evaluate the function

$$Y_{1,-1} \equiv (\boldsymbol{L}_x - i\boldsymbol{L}_y)Y_{1,0}$$

and show that it is an eigenfunction of $\boldsymbol{L}_z$ with eigenvalue $\mu = -\hbar$.

[16 marks]

The functions $Y_{1,0}$ and $Y_{1,-1}$ are eigenfunctions of which other operator?

[4 marks]

5) Consider a particle confined to the infinite square-well potential

$$V(x) = \begin{cases} 0\,, & 0 < x < L\,, \\ \infty\,, & \text{elsewhere.} \end{cases}$$

The time-independent energy eigenfunctions are of the form

$$u(x) = A \sin kx\,.$$

(i) By considering the boundary conditions, find the allowed values of k;

[5 marks]

(ii) By applying the normalisation condition, show that $A = \sqrt{2/L}$;

[5 marks]

(iii) Show explicitly that different eigenfunctions are orthogonal;

[5 marks]

(iv) By substituting the eigenfunctions into Schrödinger's equation, determine the energy eigenvalues;

[5 marks]

(v) Calculate the mean value of x for each eigenfunction;

[5 marks]

(vi) Suppose that, at a certain time, the particle is in the state

$$\psi(x) = \sqrt{\frac{1}{L}}\sin\frac{\pi x}{L} + \frac{1}{2}\sqrt{\frac{1}{L}}\sin\frac{2\pi x}{L} + \frac{1}{2}\sqrt{\frac{3}{L}}\sin\frac{4\pi x}{L}\,.$$

Determine the probability that, on measuring the particle's energy, one obtains a value corresponding to (a) the ground state, and (b) the first excited state.

[5 marks]

Note. You will find the following integrals useful:

$$\int_0^{\pi} \sin mx\, \sin nx\, \mathrm{d}x = \frac{\pi}{2}\delta_{m,n}$$

$$\int_0^{\pi} x \sin^2 mx\, \mathrm{d}x = \frac{\pi^2}{4}$$

where $m, n = 1, 2, 3, \ldots$ and $\delta_{m,n}$ is the Kronecker delta.