King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/3630 General Relativity and Cosmology

Summer 2001

Time Allowed: THREE HOURS

Candidates must answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section on the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. When necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Speed of light in free space	$c = 2.998 \times 10^8$ m s^{-1}
Newton's constant	$G_N = 6.673 \times 10^{-11}$ m^3 s^{-2} kg^{-1}
Planck's constant	$h = 6.626 \times 10^{-34}$ J s
Mass of Sun	$M_\odot = 1.989 \times 10^{30}$ kg
	$= 1.477 \times 10^3$ m
Schwarzschild metric (in units with $G_N = c = 1$)	$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$
Energy in Schwarzschild geometry	$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$
Christoffel symbols:	$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$.
Riemann Curvature Tensor (RCT):	$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\kappa\mu}\Gamma^\kappa{}_{\beta\nu} - \Gamma^\alpha{}_{\kappa\nu}\Gamma^\kappa{}_{\beta\mu}$.
Properties of RCT:	$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}$.

SECTION A : Answer SIX parts of this section

1.1) By assuming (without proof) Birkhoff's theorem for spherically-symmetric space times, determine the nature of gravitational forces experienced by test particles in the interior of a self-gravitating hollow sphere.

[7 marks]

1.2) In a coordinate system with coordinates x^μ, $\mu = 0, 1, \ldots 3$, the invariant line element is $ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$, where $\eta_{\alpha\beta}$ is the Minkowski metric, and repeated indices denote summation as usual. If the coordinates are transformed $x^\mu \to \overline{x}^\mu$, show that the line element acquires the form $ds^2 = \bar{g}_{\mu\nu} d\overline{x}^\mu d\overline{x}^\nu$, and express $\bar{g}_{\mu\nu}$ in terms of $\partial x^\mu / \partial \bar{x}^\nu$.

[7 marks]

1.3) Consider the two-dimensional metric space $ds^2 = dr^2 + r^2 d\theta^2$, where (r, θ) are polar coordinates. Write down the two geodesic equations.

[7 marks]

1.4) Consider a spherically-symmetric non-rotating body of mass M, and let two concentric shells surrounding the body be located at $r_1 = 4M$ and $r_2 = 8M$, where r denotes the radial spherical polar coordinate, and we work in a system of units in which mass is measured in units of length. Let light be emitted from the shell r_1 and absorbed at the shell r_2. Show that the period of this light is increased by a factor 1.22 as a consequence of the gravitational red-shift.

[7 marks]

1.5) Starting from rest at a great distance an observer is plunging straight (i.e. radially) towards a non-rotating black hole of mass $8M_\odot$, where $M_\odot$ is the solar mass. The observer sets his wristwatch to noon as he determines (by one means or another) that he is crossing the horizon. Determine how much time (in seconds) is left, according to the wristwatch of the observer, until the instant of crunch (i.e. when he approaches the singularity). Assume without proof the formula for the energy in the Schwarzschild geometry (see rubric on the first page), involving proper (wristwatch) and far-away times.

[7 marks]

1.6) Find the Christoffel symbols, and from these compute the Riemann curvature tensor, for the two-dimensional space time with coordinates (v, w) and metric:

$$ds^2 = dv^2 - v^2 dw^2 \tag{1}$$

What do you conclude about this space time?

[7 marks]

1.7) Let $A^{\mu\nu} = -A^{\nu\mu}$ be an antisymmetric rank-$\binom{2}{0}$ tensor, and let $S_{\mu\nu} = S_{\nu\mu}$ be a symmetric rank-$\binom{0}{2}$ tensor. Show that $S_{\mu\nu}A^{\mu\nu}$ vanishes, and that for an arbitrary rank-$\binom{2}{0}$ tensor $B^{\mu\nu}$, the following relations are true:

$$B^{\mu\nu}A_{\mu\nu} = \frac{1}{2}\Big[B^{\mu\nu} - B^{\nu\mu}\Big]A_{\mu\nu},$$
$$B^{\mu\nu}S_{\mu\nu} = \frac{1}{2}\Big[B^{\mu\nu} + B^{\nu\mu}\Big]S_{\mu\nu}.$$

[7 marks]

1.8) Assume (without proof) that the laws of conservation of total energy and angular momentum during an orbit of a satellite of mass m, and angular momentum per unit mass L/m, around a Black Hole of mass M, imply the following formula for the square of the "effective potential" per unit mass (we work for convenience in units where $c = 1 = G_N$):

$$\left(\frac{V(r)}{m}\right)^2 = \left(1 - \frac{2M}{r}\right)\left[1 + \frac{(L/m)^2}{r^2}\right] \tag{2}$$

The corresponding effective potential in the Newtonian approach is:

$$\frac{V_N}{m} = -\frac{M}{r} + \frac{(L/m)^2}{2r^2} \ ,$$

Describe qualitatively the most important differences between Newtonian planetary motion around the Sun and General-Relativistic orbits around a Black Hole. Sketch and comment on the stability of orbits.

[7 marks]

SECTION B : Answer TWO questions

2) Consider the two-dimensional spacetime described by the infinitesimal line element:

$$ds^2 = -dt^2 + a^2(t)d\theta^2,$$

where a is a function of t, called the *scale factor* for this universe.

1. By using an appropriate variational method, or otherwise, compute the Christoffel symbols for the above spacetime.

   [6 marks]

2. How many independent components does the Riemann curvature tensor have in this spacetime? Compute all independent components of this tensor.

   [6 marks]

3. Show that the components of the Ricci tensor, defined as

   $$R_{\mu\nu} = R_{\nu\mu} = R^{\alpha}{}_{\mu\alpha\nu},$$

   for this spacetime are:

   $$R_{tt} = -\frac{\ddot{a}}{a}, \qquad R_{\theta\theta} = a\ddot{a}, \qquad R_{t\theta} = R_{\theta t} = 0.$$

   [6 marks]

4. Compute the curvature scalar, $R = g^{\mu\nu}R_{\mu\nu}$, for this spacetime.

   [6 marks]

5. If $a(t) = t^2$, discuss the evolution of this two-dimensional universe.

   [6 marks]

SEE NEXT PAGE

3) Assume the equation for the stress-energy tensor in a flat space-time $T^{\mu\nu}{}_{,\nu} = 0$. Consider a *bounded* system, i.e. a system for which $T^{\mu\nu} = 0$ outside a *bounded region* $\mathcal{D}$ of *space* (not spacetime), and on the boundary itself. Prove the following results:

(i) $\frac{\partial}{\partial t} \int_{\mathcal{D}} d^3x \; T^{0\alpha} = 0$, $\alpha = 0, \ldots 3$.

Interpret this result physically.

[6 marks]

(ii) Assume the weak-field Einstein's equations in a slightly curved space time (the so-called gravitational-wave equations):

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{3}$$

where the symbol $\nabla^2 = \delta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$ denotes the usual Laplacian in Euclidean three-dimensional space, and $T_{\mu\nu} = S_{\mu\nu}(\vec{x})e^{-i\Omega t}$, with $S_{\mu\nu}(\vec{x})$ being a function only of the spatial coordinates x^i; $S_{\mu\nu} \neq 0$ only in a bounded region of space, $\mathcal{D}$, which is assumed spherical with a radius ϵ very small compared with the wavelength $2\pi/\Omega$ of the gravitational wave of frequency Ω.

Show that a solution of (3) has the form $\bar{h}_{\mu\nu} = \mathrm{Re}\left(B_{\mu\nu}(x^i)e^{-i\Omega t}\right)$, where Re denotes the real part (which for convenience can be taken at the end of the computations), and $B_{\mu\nu}$ satisfies the equation:

$$(\nabla^2 + \Omega^2)B_{\mu\nu} = -16\pi S_{\mu\nu} \tag{4}$$

Defining $J_{\mu\nu} = \int_{\mathcal{D}} d^3x S_{\mu\nu}$ show that

$$-i\Omega J^{\mu 0} e^{-i\Omega t} = \int_{\mathcal{D}} d^3x T^{\mu 0}{}_{,0} \ .$$

[8 marks]

SEE NEXT PAGE

(iii) Using the equation $T^{\mu\nu}{}_{,\nu} = 0$ and Gauss's theorem, show that

$$i\Omega J^{\mu 0} e^{-i\Omega t} = \oint T^{\mu j} n_j dS \ ,$$

where n_j is a vector normal to a surface bounding the volume $\mathcal{D}$ completely containing the source of the gravitational waves. From this show that $J^{\mu 0} = 0$.

[6 marks]

(iv) Assume the following outgoing-wave form of $B_{\mu\nu}$:

$$B_{\mu\nu} = \frac{A_{\mu\nu}}{r} e^{i\Omega r} \tag{5}$$

where $A_{\mu\nu}$ are constants, and r is the usual spherical polar radial coordinate, whose origin is chosen to be the location of the source of the gravitational wave. Integrate (4) over the three-space, assuming that $\int d^3x \Omega^2 B_{\mu\nu}$ is *negligible* compared to the other terms of the integral. Then, by virtue of Gauss's theorem, show that

$$A_{\mu\nu} = 4J_{\mu\nu} \ ,$$
$$\bar{h}_{\mu\nu} = 4\mathrm{Re}\left(\frac{J_{\mu\nu}}{r} e^{i\Omega(r-t)}\right)$$

which, thus, gives the expression for the gravitational wave generated by the source, keeping only dominant terms as $1/r$ becomes small, i.e. r is large, where the weak-field analysis applies.

[10 marks]

4) Einstein's equations for a Friedmann–Robertson–Walker (FRW) universe, with scale factor $a(t)$ assume the form:

$$\begin{aligned} -3\frac{\dot{a}^2(t)}{a^2(t)} - 3\frac{k}{a^2(t)} + \Lambda &= -8\pi G_N \rho \ , \\ -2\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} - \frac{k}{a^2(t)} + \Lambda &= 8\pi G_N p \ , \end{aligned} \tag{6}$$

where k is the usual characteristic parameter of the FRW cosmology, G_N is Newton's constant, Λ is the cosmological constant, ρ is the energy density, and p is the pressure density.

(i) Show that from these equations one can deduce the following:

$$\dot{\rho} + 3(\rho + p)\frac{\dot{a}(t)}{a(t)} = 0 \ , \tag{7}$$

$$\frac{\ddot{a}(t)}{a(t)} + \frac{4\pi G_N}{3}(\rho + 3p) = \frac{\Lambda}{3} \ . \tag{8}$$

[8 marks]

(ii) Discuss what was the problem that Einstein wanted to solve by introducing the cosmological constant term Λ into the theory.

[6 marks]

(iii) It follows from the FRW equations that

$$\frac{d}{da(t)}(\rho a^3(t)) + 3pa^2(t) = 0.$$

Assuming this without proof, show that for a matter dominated universe ('dust')

$$\rho_{dust} \propto a^{-3}(t)$$

and for a radiation dominated universe one has

$$\rho_{rad} \propto a^{-4}(t) \ .$$

[8 marks]

(iv) Consider the asymptotic limit $a(t) \to 0$, for small t, in the case $\Lambda = 0$, and show that in this case one obtains the following approximate equation for 'dust',

$$\dot{a}^2(t) \simeq (\text{const}) \times \frac{8\pi G_N}{3a(t)} \ ,$$

and from this deduce the form of $a(t)$ as a function of time, for small t, assuming an expanding universe for small t.

[8 marks]

5) (i) How are angles defined in a general-relativistic setting ?

[5 marks]

(ii) A conformal transformation of a metric is defined by $g_{\mu\nu} \to f(x^\alpha)g_{\mu\nu}$, for an arbitrary function of the coordinates $f(x^\alpha)$. Note that this transformation does *not* affect the coordinates x^α themselves. Show that this transformation preserves all angles.

[8 marks]

(iii) Show that all null curves remain null under a conformal transformation.

[7 marks]

(iv) Consider a conformally-flat metric $ds^2 = e^{2\phi}\eta_{\alpha\beta}dx^\alpha dx^\beta$, where ϕ is a function of the coordinates x^α, $\alpha, \beta \in \{0, 1, 2, 3\}$, $\eta_{\mu\nu}$ is the Minkowski metric, and repeated indices denote summation as usual. Show that the null (straight-line) *geodesics* of the Minkowski space time remain null *geodesics* for this space time.

[10 marks]

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