Solutions to 2004 MathII Exam Paper

March 30, 2006

Section A

1 (a)

$$\frac{\partial \vec r}{\partial r} = \frac{\partial}{\partial r}\left(r\cos\theta\,\hat\imath + r\sin\theta\,\hat\jmath + z\hat k\right)$$

$$= \cos\theta\,\hat\imath + \sin\theta\,\hat\jmath\,,$$

$$\left|\frac{\partial \vec r}{\partial r}\right| = h_r = \sqrt{\cos^2\theta + \sin^2\theta} = 1,$$

$$\boxed{\hat e_r = \frac{1}{h_r}\left(\frac{\partial \vec r}{\partial r}\right) = \cos\theta\,\hat\imath + \sin\theta\,\hat\jmath}\,;$$ 3

$$\frac{\partial \vec r}{\partial \theta} = -r\sin\theta\,\hat\imath + r\cos\theta\,\hat\jmath\,,$$

$$h_\theta = \sqrt{r^2\sin^2\theta + r^2\cos^2\theta} = r$$

$$\boxed{\hat e_\theta = \frac{1}{h_\theta}\left(-r\sin\theta\,\hat\imath + r\cos\theta\,\hat\jmath\right) = -\sin\theta\,\hat\imath + \cos\theta\,\hat\jmath}\,;$$ 3

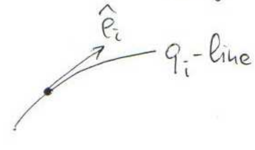
and, finally,

$$\frac{\partial \vec r}{\partial z} = \hat k\,,\quad h_z = 1,\quad \boxed{\hat e_z = \hat k}$$ 1

2 ~~grad ψ is the rate of change of ψ along~~

- $\text{grad}\,\psi = \sum_{i=1}^{3} (\text{grad}\,\psi)_i\,\hat e_i$ 1

$(\text{grad}\,\psi)_i = \dfrac{d\psi}{ds_i}$, i.e. it is the derivative $\dfrac{d\psi}{ds_i}$ in direction $\hat e_i$.



Here $d\psi = \dfrac{\partial\psi}{\partial q_i}\,dq_i$ since we move along the q_i-line, and also from 2

$$d\vec r = \sum_j h_j\,dq_j\,\hat e_j$$

follows $ds_i = h_i\,dq_i$ along this line. Therefore, 2

$$(\text{grad}\,\psi)_i = \frac{\frac{\partial\psi}{\partial q_i}dq_i}{h_i\,dq_i} = \frac{1}{h_i}\frac{\partial\psi}{\partial q_i}$$ and, thus, 2

$$\boxed{\text{grad}\,\psi = \sum_i \frac{1}{h_i}\frac{\partial\psi}{\partial q_i}\hat e_i}$$. QED

1.3

$$I = \int_{-\infty}^{\infty} e^{-2x}\left[\delta(2x+1) + 5H(x+1)\right]dx = I_\delta + I_H \qquad 1$$

$$I_\delta = \int_{-\infty}^{\infty} e^{-2x}\delta(2x+1)\,dx = \left| \begin{array}{l} y = 2x+1 \\ dy = 2dx \\ 2x = y-1 \end{array} \right| = \int_{-\infty}^{\infty} e^{-(y-1)}\delta(y)\,\frac{dy}{2}$$

$$= \frac{1}{2}e^{-(0-1)} = \frac{e}{2}; \qquad 3$$

$$I_H = \int_{-1}^{\infty} e^{-2x}\cdot 5\,dx = 5\,\frac{1}{(-2)}\,e^{-2x}\Big|_{-1}^{\infty} = \frac{5}{2}e^{2} = \frac{5e^2}{2};$$

$$I = \frac{e}{2} + \frac{5e^2}{2} = \frac{e}{2}(1+5e) \qquad 3$$

1.4 Inverse FT

$$f(t) = \int_{-\infty}^{\infty} F(\nu)\,e^{-i2\pi\nu t}\,d\nu \qquad 2$$

Conditions:

- piecewise (consists of a finite number of pieces)
- the integral $\int_{-\infty}^{\infty} |f(t)|\,dt < +\infty$ 2

Heaviside function $H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$ does not satisfy the 2nd condition: $\int_{-\infty}^{\infty} |H(t)|\,dt = \int_{0}^{\infty} dt = \infty$. 3

[However, its Fourier transform can be written via one of the generalised functions – quite formally and usefully! → not required for the student though]

(1.5) $f(t) = \begin{cases} 1, & -1 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

$$F(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{i2\pi\nu t}\, dt = \int_{-1}^{1} e^{i2\pi\nu t}\, dt = \frac{1}{i2\pi\nu} e^{i2\pi\nu t}\Big|_{-1}^{1} =$$

$$= \frac{1}{i2\pi\nu}\left[e^{i2\pi\nu} - e^{-i2\pi\nu}\right] = \frac{2i \sin 2\pi\nu}{i2\pi\nu}$$ 5

$$\boxed{= \frac{\sin 2\pi\nu}{\pi\nu}} \Rightarrow \boxed{f(t) = \int_{-\infty}^{\infty} \frac{\sin 2\pi\nu}{\pi\nu} e^{-i2\pi\nu t}\, d\nu}$$ 2

using the inverse FT.

(1.6) $$p(x) = \frac{3(x-1)}{(x^2+1)(x^2-1)^2} = \frac{3}{(x^2+1)(x-1)(x+1)^2}$$

$$q(x) = \frac{2(x+1)^2}{(x^2+1)(x^2-1)^2} = \frac{2}{(x^2+1)(x-1)^2}$$ 1

since $x^2-1 = (x-1)(x+1)$.

↳ two points:

(a) $\underline{x=1}$

$$(x-1)p(x)\Big|_{x=1} = \frac{3}{(x^2+1)(x+1)^2}\Big|_{x=1} = \text{a number, ok.}$$

$$(x-1)^2 q(x)\Big|_{x=1} = \frac{2}{(x^2+1)}\Big|_{x=1} = \text{a number, ok,}$$ 3

this point is a regular singular point (RSP)

(b) $\underline{x=-1}$

$$(x+1)p(x)\Big|_{x=-1} = \frac{3}{(x^2+1)(x-1)(x+1)}\Big|_{x=-1} \rightarrow \text{singular at } x=-1$$

$$(x+1)^2 q(x)\Big|_{x=-1} = \frac{2(x+1)^2}{(x^2+1)(x-1)^2} \rightarrow \text{a number, ok.}$$ 3

Thus, $\underline{x=-1}$ is irregular singular point.

(1.7) $y(x,t) = X(x)T(t)$ (2)

$$\frac{1}{\varkappa} X T' = X'' T$$

Divide by XT:

$$\left.\begin{array}{l} \underbrace{\frac{1}{\varkappa}\frac{T'}{T}}_{\text{depends only on } t} = \underbrace{\frac{X''}{X}}_{\text{depends only on } X} \longrightarrow \dfrac{X''}{X} = -k \\ \qquad\qquad\qquad\qquad\qquad\quad \dfrac{1}{\varkappa}\dfrac{T'}{T} = -k \end{array}\right\}$$ (3)

and we obtain (+k is also accepted)

$$X'' + kX = 0 \quad \leftarrow \text{ODE for } X(x)$$ (2)

$$T' = -k\varkappa T \quad \leftarrow \text{ODE for } T(t)$$

(1.8) $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$

$$P_0(x) = \frac{1}{2^0 0!}(x^2-1)^0 = 1$$ (1)

$$P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = \frac{1}{2}\cdot 2x = x$$ (2)

$$P_2(x) = \frac{1}{2^2 2}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{2^3}\frac{d}{dx}\left[2(x^2-1)2x\right] =$$

$$= \frac{1}{8}\frac{d}{dx}\left(4x^3-4x\right) = \frac{4}{8}\left(3x^2-1\right) = \frac{1}{2}(3x^2-1)$$ (2)

Orthogonality:

$$\int_{-1}^{1} P_0 P_2\, dx = \int_{-1}^{1} 1\cdot\frac{1}{2}(3x^2-1)\,dx = \left(\frac{\cancel{3}}{2}\frac{x^3}{\cancel{3}} - \frac{x}{2}\right)\Bigg|_{-1}^{1} = 0$$ (1)

$$\int_{-1}^{1} P_1 P_2\, dx = \frac{1}{2}\int_{-1}^{1}(3x^3-x)\,dx = \frac{1}{2}\left[\frac{3x^4}{4} - \frac{x^2}{2}\right]_{-1}^{1} = 0$$ as required. (1

Section B

2

a) Solving for $\hat{\imath}, \hat{\jmath}, \hat{k}$ in the equations given in problem A1.1, we obtain:

$$\begin{cases} \hat{\imath} = \cos\theta\, \hat{e}_r - \sin\theta\, \hat{e}_\theta \\ \hat{\jmath} = \sin\theta\, \hat{e}_r + \cos\theta\, \hat{e}_\theta \\ \hat{k} = \hat{e}_z \end{cases}$$

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b) Differentiate $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$ wrt time t using problem A1.1, and then substitute back $\hat{\imath}, \hat{\jmath}, \hat{k}$ via $\hat{e}_i$ from ~~the~~ (a):

$$\frac{d\hat{e}_r}{dt} = -\sin\theta \cdot \dot{\theta}\, \hat{\imath} + \cos\theta \cdot \dot{\theta}\, \hat{\jmath} = \dot{\theta}\Big[-\sin\theta\left(\cos\theta\, \hat{e}_r - \sin\theta\, \hat{e}_\theta\right) +$$

$$+ \cos\theta\left(\sin\theta\, \hat{e}_r + \cos\theta\, \hat{e}_\theta\right)\Big] = \dot{\theta}\Big[\hat{e}_r\left(-\sin\theta\cos\theta + \cos\theta\sin\theta\right) +$$

$$+ \hat{e}_\theta\left(\sin^2\theta + \cos^2\theta\right)\Big] = \dot{\theta}\, \hat{e}_\theta \;;$$

3

$$\frac{d\hat{e}_\theta}{dt} = -\cos\theta \cdot \dot{\theta}\, \hat{\imath} - \sin\theta \cdot \dot{\theta}\, \hat{\jmath} =$$

$$= -\dot{\theta}\Big[\cos\theta\left(\cos\theta\, \hat{e}_r - \sin\theta\, \hat{e}_\theta\right) + \sin\theta\left(\sin\theta\, \hat{e}_r + \cos\theta\, \hat{e}_\theta\right)\Big]$$

$$= -\dot{\theta}\Big[\hat{e}_r\left(\cos^2\theta + \sin^2\theta\right) + \hat{e}_\theta\left(-\cos\theta\sin\theta + \sin\theta\cos\theta\right)\Big]$$

$$= -\dot{\theta}\, \hat{e}_r \;;$$

3

$$\frac{d\hat{e}_z}{dt} = 0$$

1

$\boxed{c}$ $d\bar{r} = \bar{r}(q_1+dq_1, q_2+dq_2, q_3+dq_3) - \bar{r}(q_1 q_2 q_3)$

$= \sum_i \frac{\partial \bar{r}}{\partial q_i} dq_i \equiv \sum_i \hat{e}_i dq_i h_i$, Q.E.D.

4

$\boxed{d}$ From ~~problem A 1.1~~ above:

$d\vec{r} = h_1 dq_1 \hat{e}_1 + h_2 dq_2 \hat{e}_2 + h_3 dq_3 \hat{e}_3$.

or $d\bar{r} = h_r dr \hat{e}_r + h_\theta d\theta \hat{e}_\theta + h_z dz \hat{e}_z$

$= \hat{e}_r dr + \hat{e}_\theta r d\theta + e_z dz$

and, thus, the velocity

$\vec{V} = \frac{d\bar{r}}{dt} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{e}_z$

3

• The acceleration

$\bar{a} = \frac{d\bar{v}}{dt} = \ddot{r}\hat{e}_r + \dot{r}(\dot{\hat{e}}_r) + (r\dot{\theta})'\hat{e}_\theta + r\dot{\theta}(\dot{\hat{e}}_\theta) + \ddot{z}\hat{e}_z$

$= \ddot{r}\hat{e}_r + \dot{r}(\dot{\theta}\hat{e}_\theta) + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta + r\dot{\theta}(-\dot{\theta}\hat{e}_r) + \ddot{z}\hat{e}_z$

$= (\ddot{r} - r\dot{\theta}^2)e_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta + \ddot{z}\hat{e}_z$

5

$\boxed{e}$ Eqs. of motion along $\hat{e}_r$ and $\hat{e}_\theta$ follow from the Newtonian's eqs. of motion:

$\bar{F} = m\vec{a}$ with $\vec{a}$ from

the previous part and $m=1$. Thus:

$\boxed{\ddot{r} - r\dot{\theta}^2 = f(r)}$ and $\boxed{2\dot{r}\dot{\theta} + r\ddot{\theta} = 0}$

4

For a circular trajectory $\dot{r} = 0$, and from the 2nd eq. of motion $\ddot{\theta} = 0 \rightarrow \boxed{\theta = \theta_0 + \omega t}$, where $\omega = \dot{\theta}(0)$, $\theta_0 = \theta(0)$.

2

3

(a) $\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{i2\pi\nu t}\, dt = e^{i2\pi\nu \cdot 0} = 1 \equiv F(\nu)$. 2

Using the inverse FT:

$\cancel{\mathcal{F}^{-1}[\delta(t)]}\ \delta(t) \equiv \int_{-\infty}^{\infty} 1 \cdot e^{-i2\pi\nu t}\, d\nu = \int_{-\infty} e^{-i2\pi\nu t}\, d\nu$ 3

(b) $F(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{i2\pi\nu t}\, dt$ 1

$G(\nu) = \int_{-\infty}^{\infty} g(t)\, e^{i2\pi\nu t}\, dt$

Using these in the integral

$\int_{-\infty}^{\infty} F(\nu)\, G(\nu)^{*}\, d\nu = \int_{-\infty}^{\infty} d\nu \left[\int_{-\infty}^{\infty} f(t)\, e^{i2\pi\nu t}\, dt \right] \left[\int_{-\infty}^{\infty} g(t')\, e^{-i2\pi\nu t'}\, dt' \right]$ 3

$= \int dt\, dt'\, f(t)\, g(t') \underbrace{\left[\int_{-\infty}^{\infty} d\nu\, e^{i2\pi\nu(t-t')} \right]}_{\delta(t-t')}$

$= \int dt\, dt'\, f(t)\, g(t)\, \delta(t-t') = \int dt\, f(t)\, g(t)$ 3

after integrating e.g. with respect to t.

(b) By definition,

$F(\nu) = \int_{-\infty}^{\infty} \rho(t)\, e^{i2\pi\nu t}\, dt = \int_{-\infty}^{\infty} dt\, e^{i2\pi\nu t} \int_{-\infty}^{\infty} d\tau\, f(t-\tau)\, g(\tau)$

$= \int d\tau\, g(\tau)\, e^{i2\pi\nu\tau} \underbrace{\int dt\, f(t-\tau)\, e^{i2\pi\nu(t-\tau)}}_{F(\nu)}$ 6

$= F(\nu) \int d\tau\, g(\tau)\, e^{i2\pi\nu\tau} = F(\nu)\, G(\nu)$

(c) $f(t) = \begin{cases} e^{-\alpha t}, & t \geq 0 \\ 0, & t<0 \end{cases}$

$$f*f = \int_{-\infty}^{\infty} f(t-\tau)\, f(\tau)\, d\tau$$

Only positive argument of f contributes:

$$\tau \geq 0, \qquad t-\tau \geq 0 \to \tau \leq t$$

Assuming $t>0$:

$$f*f = \int_0^t f(t-\tau)\, f(\tau)\, d\tau = \int_0^t e^{-\alpha(t-\tau)} e^{-\alpha\tau}\, d\tau =$$

$$= \int_0^t e^{-\alpha t}\, d\tau = e^{-\alpha t}\int_0^t d\tau = t e^{-\alpha t}, \quad t\geq 0$$ 3

If $t<0$ then conditions $\tau\geq 0$, $\tau\leq t<0$ contradict each other, so that for $t<0$ ~~the arguments of $f(t-\tau)$ and $f(\tau)$ are~~ τ should be negative when $f(\tau)\equiv 0 \to f*f = 0$. Thus, 2

$$f*f = \begin{cases} t e^{-\alpha t}, & t\geq 0 \\ 0, & t<0 \end{cases} = t f(t).$$ 1

(d) • FT of $f(t)$ is

$$F(\nu) = \int f(t)\, e^{i2\pi\nu t}\, dt = \int_0^\infty e^{-\alpha t} e^{i2\pi\nu t}\, dt = \frac{1}{i2\pi\nu-\alpha} e^{(-\alpha+i2\pi\nu)t}\Big|_0^\infty$$

$$= -\frac{1}{i2\pi\nu-\alpha} = \frac{1}{\alpha-i2\pi\nu}$$ 3

• FT of $d(t) = t f(t)$ is:

$$D(\nu) = \int_0^\infty t e^{-\alpha t} e^{i2\pi\nu t}\, dt = (\text{by parts}) = \underbrace{\left[t\,\frac{e^{(-\alpha+i2\pi\nu)t}}{-\alpha+i2\pi\nu}\right]\Big|_0^\infty}_{0} -$$

$$- \int_0^\infty \frac{e^{(-\alpha+i2\pi\nu)t}}{-\alpha+i2\pi\nu}\, dt = -\frac{1}{(-\alpha+i2\pi\nu)^2}\, e^{(-\alpha+i2\pi\nu)t}\Big|_0^\infty =$$ 5

-9-

$= (-\alpha + i2\pi\nu)^{-2}$.

(e) $D(\nu) = (\alpha - i2\pi\nu)^{-2} = (\alpha - i2\pi\nu)^{-1} \cdot (\alpha - i2\pi\nu)^{-1} \equiv F(\nu)^2$,

where $\mathcal{F}^{-1}[F(\nu)] = f(t) \equiv \begin{cases} e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

Therefore, according to the convolution theorem,

$$\mathcal{F}^{-1}[D(\nu)] = d(t) = \int_{-\infty}^{\infty} \underbrace{f(t-\tau)}_{\neq 0 \text{ only for } t > \tau, \text{ i.e. } \tau < t} \underbrace{f(\tau)}_{\neq 0 \text{ only for } \tau > 0} d\tau \equiv f * f \equiv d(t)$$

$\cancel{\int_{-\infty}^{\infty}}$

Question 4 $\quad 36x^2y'' + (5-9x^2)y = 0$ $\quad$ -10-

ⓐ $x=0$ is the only singular point:

$p(x) = 0$

$q(x) = \dfrac{5-9x^2}{36x^2} = \dfrac{5}{36x^2} - \dfrac{9}{36}$ $\quad$ (2)

$\left.\begin{array}{l} xp \to \text{regular at } x=0 \\ x^2q = \dfrac{5}{36} - \dfrac{9}{36}x^2 \to \text{also regular at } x=0 \end{array}\right\} x=0 \text{ is a } \underline{\underline{RSP}}$

ⓑ • Seeking the solution in the form:

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n,$$ (2)

we obtain:

$$y' = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s-2}$$

• Substitute into the equation:

$$\sum_{n=0}^{\infty} \left[36 a_n (n+s)(n+s-1) x^{n+s} + (5-9x^2) a_n x^{n+s} \right] = 0$$

$$\sum_{n=0}^{\infty} [36(n+s)(n+s-1)+5] a_n x^{n+s} - \sum_{n=0}^{\infty} 9 a_n x^{n+s+2} = 0$$

• Separate out the first two terms in the first summation:

$$[36s(s-1)+5] a_0 x^s + [36(s+1)s+5] a_1 x^{s+1}$$

$$+ \sum_{n=2}^{\infty} [36(n+s)(n+s-1)+5] a_n x^{n+s} - \sum_{n=0}^{\infty} 9 a_n x^{n+s+2} = 0$$

In the last term: $n \to n+2$ and then combine with the preceding term:

$$[36s(s-1)+5] a_0 x^s + [36s(s+1)+5] a_1 x^{s+1} + \sum_{n=2}^{\infty} \Big\{ [36(n+s)(n+s-1)+5] a_n -$$ (5)

$$- 9 a_{n-2} \Big\} x^{n+s} = 0 \qquad (*)$$

• The $\underline{\text{indicial equation}}$ ~~might~~ may be either coming from the first ~~or if~~

or the second term; either is correct. Use the first one:

$\boxed{36s(s-1)+5=0} \longrightarrow \cancel{\quad}\ 36s^2-36s+5=0$ 2

$s^2 - s + \frac{5}{36} = 0$, $\underline{\underline{s_1 = \frac{5}{6}}}$, $\underline{\underline{s_2 = \frac{1}{6}}}$

The recurrence relation follows from the last term in (*):

$$\left[36(n+s)(n+s-1)+5\right] a_n = 9\, a_{n-2},$$

$$a_n = \frac{9}{36(n+s)(n+s-1)+5}\, a_{n-2}, \qquad n \geqslant 2$$ 3 (**)

$\boxed{c}$ Consider the $\underline{\text{first solution}}$, $y_1(x)$, corresponding to $s_1 = \frac{5}{6}$.

a_0 – arbitrary (by construction) 2

$36s(s+1)+5 \neq 0 \longrightarrow \boxed{a_1 \equiv 0}$ in the second term in (*)

Then, $$a_n = \frac{9}{36\left(n+\frac{5}{6}\right)\left(n-\frac{1}{6}\right)+5}\, a_{n-2} = \frac{9}{(6n+5)(6n-1)+5}\, a_{n-2}$$ 1

• The first three terms follows at once:

$\boxed{n=2} \to a_2 = \frac{9}{(6\cdot2+5)(6\cdot2-1)+5}\, a_0 = \frac{9a_0}{17\cdot11+5} = \frac{9a_0}{192} = \frac{3a_0}{64}$ 1

$n=3 \to a_3 = \frac{\cdots}{\cdots}\, a_1 = 0$ since $a_1 = 0$; only even $a_n \neq 0$!

$n=4 \to a_4 = \frac{9\, a_2}{(6\cdot4+5)(6\cdot4-1)+5} = \frac{9\cdot a_2}{29\cdot23+5} = \frac{9a_2}{672} = \frac{3a_2}{224}$

$= \frac{3}{224}\cdot\frac{3}{64}\, a_0 = \frac{3^2}{14336}\, a_0$ 1

$n=5 \to a_5 = 0$, etc. 1

Dropping an arbitrary a_0, we arrive at the first solution:

$$\boxed{y_1(x) = x^{5/6}\left[1 + \frac{3}{64}x^2 + \frac{9}{14336}x^4 + \ldots\right]}$$ 1

• Similarly for the $\underline{\text{2nd solution}}$: $s = 1/6$

a_0 – arbitrary 1

From the 2nd term in (*):

$36\,\frac{1}{6}\left(\frac{1}{6}+1\right)+5 \neq 0 \;\rightarrow\; \boxed{a_1 = 0}$ 1

The recurrence relation (**) reads in this case:

$$a_n = \frac{9}{36\left(\frac{1}{6}+n\right)\left(n-\frac{5}{6}\right)+5}\,a_{n-2} = \frac{9}{(6n+1)(6n-5)+5}\,a_{n-2}\quad,\quad n\geq 2$$ 1

- Therefore, the 1st 3 terms:

$n=2 \rightarrow a_2 = \frac{9}{13\cdot 7+5}\,a_0 = \frac{9a_0}{96} = \frac{3a_0}{32}$, 1

$n=3 \rightarrow a_3 = \frac{\cdots}{\cdots}\,a_1 \equiv 0$, all odd terms vanish!

$n=4 \rightarrow a_4 = \frac{9}{25\cdot 19+5}\,a_2 = \frac{9}{480}\,a_2 = \frac{3}{160}\,a_2 = \frac{3}{160}\cdot\frac{3}{32}\,a_0 =$

$= \frac{9}{5120}\,a_0$ 1

$n=5 \rightarrow a_5 = 0$, etc. 1

Dropping arbitrary a_0, we obtain

$$\boxed{y_2(x) = x^{1/6}\left[1+\frac{3}{32}x^2+\frac{9}{5120}x^4+\ldots\right]}$$ 1

(d) Hence, the general solution

$$y(x) = A\,y_1(x) + B\,y_2(x)$$ 2

A, B - arbitrary constants.

(5) (a) The problem has the spherical symmetry → use spherical coordinates ~~with~~ (r, θ, ϕ). In addition, the temperature will only depend on r, not on θ, ϕ. Then, the r-part of the Laplacian ($h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$) is

$$\Delta u = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial r}\left(\frac{r^2\sin\theta}{1}\frac{\partial u}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)$$

We do not need θ, ϕ-parts of Δ as u does not depend on these variables. 5

Thus, our equation is simplified into:

$$\frac{1}{\mu^2}\frac{\partial u}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) \equiv \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}, \text{ QED.}$$

(b) The temperature in the sphere be that of the water eventually, i.e. $u(r,\infty) = 10$ degrees. 1

(c) The partial differential eq. for $v(r,t)$ is the same as for $u(r,t)$, i.e.

$$\frac{1}{\mu^2}\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r}$$ 1

Boundary condition: $v(a,t) = \cancel{10}\ 0$ 1

Initial condition: $v(r,0) = -10$. 1

(d) Using $v(r,t) = R(r)T(t)$, we obtain:

$$\frac{1}{\mu^2}RT' = TR'' + \frac{2}{r}TR'$$

Divide by RT: $$\underbrace{\frac{1}{\mu^2}\frac{T'}{T}}_{\text{depends on }t} = \underbrace{\frac{R''}{R} + \frac{2}{r}\frac{R'}{R}}_{\text{on } r \text{ only}}$$

Introduce a negative seperation constant $-k^2$:

$$\frac{1}{\mu^2}\frac{T'}{T} = -k^2 \quad\text{and}\quad \frac{R''}{R} + \frac{2}{r}\frac{R'}{R} = -k^2$$ 5

which are the required equations.

$T' = -(k\varkappa)^2 T$ and $R'' + \frac{2}{r}R' + k^2 R = 0$

(d). Checking for $T(t) = \exp(-(\varkappa k)^2 t)$:

$T' = T\,(-\varkappa k)^2$ as required. (1)

- Checking for $R(r) = \frac{\sin kr}{r}$: (1)

$R' = k\frac{\cos kr}{r} - \frac{\sin kr}{r^2}$, $R'' = -2k\frac{\cos kr}{r^2} + \frac{2\sin kr}{r^3} - k^2\frac{\sin kr}{r}$

After substitution, we convince ourselves that it is a solution

- The separation constant is negative, $-k^2$, since otherwise $T(t)$ would indefinitely increase with time which is unphysical. (2)

(e) Boundary condition: $R(a) = 0 \rightarrow \frac{\sin ka}{a} = 0$,

$ka = \pi n$, $n = 0, \pm 1, \pm 2, \ldots$

However, $n = 0 \rightarrow k = 0$, which cannot be (trivially, $R \equiv 0$ in this case). Further, negative n give equivalent solutions to the positive ones. Therefore, (4)

$n = 1, 2, 3, \ldots$

(f) Hence, using a linear combination of all possible elementary solutions, each satisfying the boundary condition (superposition principle), we obtain

$$v(r,t) = \frac{1}{r}\sum_{n=1}^{\infty} v_n\, e^{-(k_n\varkappa)^2 t} \sin k_n r, \quad k_n = \frac{\pi n}{a} \quad (2)$$

(g) Applying initial conditions to obtain v_n:

$$-10 = \frac{1}{r}\sum_{n=1}^{\infty} v_n \sin k_n r \equiv \frac{1}{r}\sum_{n=1}^{\infty} v_n \phi_n(r), \quad \phi_n(r) = \sin k_n r \quad (1)$$

Functions $\phi_n(r)$ are orthogonal, so that we obtain:

$$-10r = \sum_n v_n \phi_n(r) \rightarrow v_n = -\frac{2}{a}\int_0^a 10r\,\phi_n(r)\,dr \quad (2)$$

$$= \frac{20}{a}\int_0^a r\sin k_n r\,dr = \left(\text{by parts}\right) =$$

$$= \frac{20}{a}\left[-r\,\frac{\cos k_n r}{k_n}\Big|_0^a + \frac{1}{k_n}\int_0^a \cos k_n r\,dr\right]$$

$$= \frac{10}{a}\left[-\frac{a}{k_n}\underbrace{\cos k_n a}_{\cos \pi n = (-1)^n} + \frac{1}{k_n^2}\underbrace{\sin k_n r\Big|_0^a}_{=0}\right] = \frac{20}{k_n}(-1)^n$$

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Therefore, the final solution is

$$\boxed{u(r,t) = 10 + \frac{20a}{\pi r}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-(\varkappa k_n)^2 t}\sin k_n r, \quad k_n = \frac{\pi n}{a}}$$

It satisfies both initial and boundary conditions. It tends to 10 at $t \to \infty$ as required.