Solutions to 2003 MathII Exam Paper

March 30, 2006

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Section A

(1.1) Consider a change of

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

1

along a q_1-line (q_2, q_3 - fixed): the partial derivative

$$\frac{\partial \bar{r}}{\partial q_1} = \frac{\partial x}{\partial q_1}\hat{i} + \frac{\partial y}{\partial q_1}\hat{j} + \frac{\partial z}{\partial q_1}\hat{k} \equiv h_1 \bar{e}_1$$

2

will be directed along $\bar{e}_1$, where

$$h_1 = \left|\frac{\partial \bar{r}}{\partial q_1}\right| = \sqrt{\left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial z}{\partial q_1}\right)^2}$$

4

is the scale factor. introduced to make $\hat{e}_1$ of length one.

Similarly for $\bar{e}_2, h_2$ and $\bar{e}_3, h_3$:

$$h_i = \left|\frac{\partial \bar{r}}{\partial q_i}\right| \text{ and } \bar{e}_i = \frac{1}{h_i}\frac{\partial \bar{r}}{\partial q_i}$$

(1.2)

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

Using general formulae from question 1.1:

(a)

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi, \quad \frac{\partial y}{\partial r} = \sin\theta\sin\phi, \quad \frac{\partial z}{\partial r} = \cos\theta$$

3

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi, \quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi, \quad \frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi, \quad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi, \quad \frac{\partial z}{\partial \phi} = 0$$

(b) Therefore,

2 $$\begin{cases} h_r = \left|\dfrac{\partial \vec{r}}{\partial r}\right| = \sqrt{(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2} = 1 \\ h_\theta = \sqrt{(r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2} = r \\ h_\phi = \sqrt{(-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2 + 0^2} = r\sin\theta \end{cases}$$

(c) and thus

2 $$\begin{cases} \bar{e}_r = \dfrac{1}{h_r}\left[\dfrac{\partial x}{\partial r}\hat{i} + \dfrac{\partial y}{\partial r}\hat{j} + \dfrac{\partial z}{\partial r}\hat{k}\right] = \sin\theta(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) + \\ \qquad + \cos\theta\,\hat{k}\ ; \\ \bar{e}_\theta = \cos\theta(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) - \sin\theta\,\hat{k} \\ \bar{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} \end{cases}$$

1,3

$$\int_{-\infty}^{\infty} \delta\left(\frac{x}{5} - 1\right) f(x)\,dx = \left| \begin{array}{l} \text{change variable} \\ y = \dfrac{x}{5} - 1 \end{array} \right|$$

7 $$= 5\int_{-\infty}^{\infty} \delta(y)\, f(5y + 5)\, dy = 5 f(5)$$

(1.4) • formal definition of the integral Fourier transform

2 $$F(\nu) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\nu t}\, dt$$

~~[+ sign in the exponential is also acceptable]~~

• Inverse transform:

2 $$f(t) = \mathcal{F}^{-1}[F(\nu)] = \int_{-\infty}^{\infty} F(\nu)\, e^{+i2\pi\nu t}\, d\nu$$

~~[- sign is acceptable if + was chosen in F(ν)]~~

[an opposite choice of signs in the exponentials is also perfectly acceptable]

• Consider

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-i2\pi\nu t}\, dt = 1,$$

3 so that the inverse transform

$$\delta(t) = \int_{-\infty}^{\infty} e^{i2\pi\nu t}\, d\nu, \quad \text{Q.E.D.}$$

(1.5) 1 $f(t) = e^{-\alpha|t|} = \begin{cases} e^{-\alpha t}, & t \geq 0 \\ e^{\alpha t}, & t \leq 0 \end{cases}$

Its Fourier transform

$$F(\nu) = \int_{-\infty}^{\infty} e^{-\alpha|t|} e^{-i2\pi\nu t} dt = \int_{-\infty}^{0} e^{\alpha t} e^{-i2\pi\nu t} dt +$$

$$+ \int_{0}^{\infty} e^{-\alpha t} e^{-i2\pi\nu t} dt = \frac{1}{\alpha - i2\pi\nu} e^{(\alpha - i2\pi\nu)t} \Big|_{-\infty}^{0} +$$

$$+ \frac{1}{-\alpha - i2\pi\nu} e^{-(\alpha + i2\pi\nu)t} \Big|_{0}^{\infty} =$$

6 $$= \frac{1}{\alpha - i2\pi\nu} + \frac{1}{\alpha + i2\pi\nu} = \frac{2\alpha}{\alpha^2 + (2\pi\nu)^2}$$

(1.6) Rewrite equation by dividing on $(x^2-1)(x-4)$ as:

1 $$y''(x) + \underbrace{\frac{1}{(x+1)(x-4)}}_{p(x)} y'(x) + \underbrace{\frac{1}{(x-1)(x-4)}}_{q(x)} y(x) = 0$$

There are three singular points:

(a) $x = -1$

2 $\left.\begin{array}{l} (x+1)p(x) \text{ is regular} \\ (x+1)^2 q(x) \text{ is regular} \end{array}\right] \to$ regular singular point

(b) $x = +1$

$(x-1)p(x), (x-1)^2 q(x)$ are both regular at $x = 1$

2 $\mapsto$ also is a regular singular point

(c) $\underline{x=4}$

2 again, since both $(x-4)p(x)$ and $(x-4)^2 q(x)$ are regular at this point $\to$ it is a regular singular point.

(1.7) First, checking $y_1(x+ct) \equiv y_1(u)$:

$$\frac{\partial y_1}{\partial x} = \frac{dy_1}{du}, \quad \frac{\partial^2 y_1}{\partial x^2} = \frac{d^2 y_1}{du} \quad \text{since } \frac{\partial u}{\partial x} = 1;$$

3 $$\frac{\partial y_1}{\partial t} = \frac{dy_1}{du}\frac{\partial u}{\partial t} = \frac{dy_1}{du}\cdot c, \quad \frac{\partial^2 y_1}{\partial t^2} = c^2 \frac{dy_1}{du^2},$$

so that $\partial y_1 / \partial t^2 = c^2 \cdot \frac{\partial^2 y_1}{\partial x^2}$, i.e. it is a solution.

Similarly for $y_2(x,t) = y_2(x-ct) \equiv y_2(v)$:

$$\frac{\partial y_2}{\partial x} = \frac{dy_2}{dv}, \quad \frac{\partial^2 y_2}{\partial x^2} = \frac{d^2 y_2}{dv^2}$$

3 $$\frac{\partial y_2}{\partial t} = -c\frac{dy_2}{dv}, \quad \frac{\partial^2 y_2}{\partial t^2} = (-c)^2 \frac{d^2 y_2}{dv^2} = c^2 \frac{d^2 y_2}{dv^2} \equiv c^2 \frac{\partial^2 y_2}{\partial x^2}$$

Q.E.D.

1 ~~The~~ sum of two solutions is always a solution. (a superposition principle)

(1.8) • Definition of the Legendre polinomials based on the generating function is as follows:

1 $$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

2 • Therefore, what is needed is to expand $G(x,t)$ in the Maclauren series with respect to t for $n=0$ and 1:

$$G(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!}\, G_t^{(n)}(x,t)\Big|_{t=0} \cdot t^n =$$

~~In particular,~~ $= G(x,0) + G'(x,0)\cdot t + \ldots$

$\equiv P_0(x) + P_1(x)\, t + \ldots$,

so that

$\frac{1}{2}$ $P_0(x) \equiv G(x,0) = 1$;

$\frac{1}{2}$ $$P_1(x) \equiv G'(x,0) = \left(\frac{d}{dt}\frac{1}{\sqrt{1-2xt+t^2}}\right)_{t=0} = -\frac{1}{2}\,\frac{-2x+2t}{\sqrt{\ldots}}\Bigg|_{t=0} =$$

$$= -\frac{1}{2}\,\frac{-2x}{1} = x$$

Section B

(2) The parabolic coordinates (u, v, θ):

$$x = uv\cos\theta, \quad y = uv\sin\theta, \quad z = \frac{1}{2}(u^2 - v^2)$$

(a) Working out the scale factors:

$$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

2 $$h_u = \left|\frac{\partial \bar{r}}{\partial u}\right| = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2} =$$

$$= \sqrt{(v\cos\theta)^2 + (v\sin\theta)^2 + u^2} = \sqrt{v^2 + u^2},$$

2 $$h_v = \left|\frac{\partial \bar{r}}{\partial v}\right| = \sqrt{(u\cos\theta)^2 + (u\sin\theta)^2 + v^2} = \sqrt{u^2 + v^2},$$

2 $$h_\theta = \left|\frac{\partial \bar{r}}{\partial \theta}\right| = \sqrt{(-uv\sin\theta)^2 + (uv\cos\theta)^2} = uv$$

(b) Unit base vectors:

2 $$\bar{e}_u = \frac{1}{h_u}\frac{\partial \bar{r}}{\partial u} = \frac{1}{\sqrt{u^2+v^2}}\left[v(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) + u\hat{k}\right],$$

2 $$\bar{e}_v = \frac{1}{h_v}\frac{\partial \bar{r}}{\partial v} = \frac{1}{\sqrt{u^2+v^2}}\left[u(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) - v\hat{k}\right],$$

2 $$\bar{e}_\theta = \frac{1}{h_\theta}\frac{\partial \bar{r}}{\partial \theta} = \frac{1}{uv}\left[uv(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})\right] =$$

$$= -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}.$$

(c) Checking ~~the~~ the orthogonality of $\bar{e}_u, \bar{e}_v, \bar{e}_\theta$:

$$\bar{e}_u \cdot \bar{e}_v = \frac{1}{u^2+v^2}\left[uv\left(\cos^2\theta + \sin^2\theta\right) - uv\right] = 0,$$

3 $$\bar{e}_u \cdot \bar{e}_\theta = \frac{1}{\sqrt{u^2+v^2}} \cdot \frac{1}{uv}\left[uv^2\left(-\cos\theta\sin\theta + \sin\theta\cos\theta\right)\right] = 0,$$

$$\bar{e}_v \cdot \bar{e}_\theta = \frac{1}{\sqrt{u^2+v^2}} \cdot \frac{1}{uv}\left[u^2 \cdot v\left(-\cos\theta\sin\theta + \sin\theta\cos\theta\right)\right] = 0$$

1 It is orthogonal indeed.

(d) We should simply apply the result given in the case of the parabolic coordinates:

3 and $q_1 \to u, \quad q_2 \to v, \quad q_3 \to \theta$

$h_1 = h_2 = \sqrt{u^2+v^2}, \quad h_3 = uv$

so that we obtain:

$$\nabla^2\psi = \frac{1}{(u^2+v^2)uv}\left[\frac{\partial}{\partial u}\left(uv\frac{\partial\psi}{\partial u}\right) + \frac{\partial}{\partial v}\left(uv\frac{\partial\psi}{\partial v}\right) + \right.$$

3 $$\left. + \frac{\partial}{\partial\theta}\left(\frac{u^2+v^2}{uv}\frac{\partial\psi}{\partial\theta}\right)\right] =$$

1 $$= \frac{1}{uv(u^2+v^2)}\left[v\frac{\partial}{\partial u}\left(u\frac{\partial\psi}{\partial u}\right) + u\frac{\partial}{\partial v}\left(v\frac{\partial\psi}{\partial v}\right)\right] + \frac{1}{(uv)^2}\frac{\partial^2\psi}{\partial\theta^2}$$

1 or, cancelling ~~on~~ $1/uv$, we obtain the desired result.

(3)

(a) using the definition of the Fourier transform:

$$\mathcal{F}\left[f(t)\cos(2\pi\nu_0 t)\right] = \int_{-\infty}^{\infty} f(t)\cos(2\pi\nu_0 t)\, e^{-i2\pi\nu t}\, dt =$$

1 $$= \frac{1}{2}\int_{-\infty}^{\infty} f(t)\left[e^{i2\pi\nu_0 t} + e^{-i2\pi\nu_0 t}\right] e^{-i2\pi\nu t}\, dt$$

1 $$= \frac{1}{2}\int f(t)\, e^{-i2\pi(\nu-\nu_0)t}\, dt + \frac{1}{2}\int f(t)\, e^{-i2\pi(\nu+\nu_0)t}\, dt$$

4 $$\equiv \frac{1}{2}\left[F(\nu-\nu_0) + F(\nu+\nu_0)\right]$$

(b). Using integration by parts, we have:

$$\mathcal{F}[f'(t)] = \int_{-\infty}^{\infty} f'(t)\, e^{-i2\pi\nu t}\, dt =$$

$$= f(t)\, e^{-i2\pi\nu t}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\nu t}(-i2\pi\nu)\, dt$$

4 $$= (i2\pi\nu)\int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\nu t}\, dt \equiv (i2\pi\nu)\, F(\nu)$$

2 since $f(\pm\infty) = 0$, ~~which is required by the existance of the Fourier transform.~~

- To calculate $\mathcal{F}[f''(t)]$ there is no need to go through the same again, since the above result is valid for any function. In particular

$f''(t) = \left(f'(t)\right)'$, so that

4 $\mathcal{F}[f''(t)] = (i2\pi\nu)\,\mathcal{F}[f'(t)] = (i2\pi\nu)^2 F(\nu)$

(c) Consider

$$F(\nu) = \int f(t)\, e^{-i2\pi\nu t}\, dt$$

Then,

$$F'(\nu) = \int_{-\infty}^{\infty} f(t)(-i2\pi t)\, e^{-i2\pi\nu t}\, dt$$

2 $$F'(0) = (-i2\pi)^1 \int_{-\infty}^{\infty} f(t)\, t\, dt \equiv (-2\pi i)\, \mu_1$$

Similarly

$$F''(\nu) = (-2\pi i)^2 \int_{-\infty}^{\infty} f(t)\, t^2 e^{-i2\pi\nu t}\, dt$$

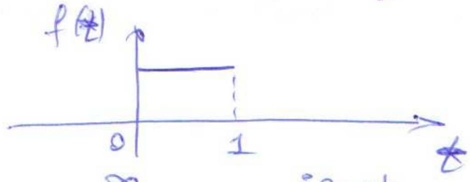
2 $$F''(0) = (-2\pi i)^2 \int_{-\infty}^{\infty} f(t)\, t^2 dt = (-2\pi i)^2 \mu_2$$

In general, it is easily seen that

4 $$F^{(n)}(\nu) = (-2\pi i)^n \int_{-\infty}^{\infty} f(t)\, t^n e^{-i2\pi\nu t}\, dt$$

and $F^{(n)}(0) = (-2\pi i)^n \mu_n$. Q.E.D.

(d) $f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t < 0 \text{ or } t > 1 \end{cases}$



$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\nu t}\, dt =$$

$$= \int_0^1 e^{-i2\pi\nu t}\, dt = \frac{1}{-i2\pi\nu}\, e^{-i2\pi\nu t}\Big|_0^1 =$$

$$= \frac{1}{-i2\pi\nu}\left(e^{-i2\pi\nu} - 1\right) = \frac{1}{i2\pi\nu}\left(1 - e^{-i2\pi\nu}\right)$$

(4)

(a) We rewrite the DE as follows:

1 $y'' + \frac{1}{x} y' + \left(1 - \frac{p^2}{x^2}\right) y = 0$

1 There is one singular point $x=0$ which is a regular singular point since

$$\underbrace{x \frac{1}{x}}_{p(x)} = 1 \text{ and } \underbrace{x^2\left(1 - \frac{p^2}{x^2}\right)}_{q(x)} = x^2 - p^2$$

2 are both regular at $x=0$.

(b) To solve the equation, we seek a solution in the

3 form: $y(x) = x^s \sum_{n=0}^{\infty} a_n x^n$

- $y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}$

2 $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$

- Substituting into the DE, we obtain:

$$\sum_{n=0}^{\infty} a_n \left[(n+s)(n+s-1) + (n+s) \right] x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} - p^2 \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

3 or $\sum_{n=0}^{\infty} a_n \left[(n+s)^2 - p^2 \right] x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0$

- The summation index in the 2nd term is changed: $n \to n+2$, while the terms $n=0$ and $n=1$ are

4 separated out in the first sum:

2 $$a_0(s^2-p^2)x^s + a_1\left[(s+1)^2-p^2\right]x^{s+1} + \sum_{n=2}^{\infty}\left\{a_n\left[(n+s)^2-p^2\right] + a_{n-2}\right\}x^{n+s} = 0$$

- The indicial equation reads

$$s^2-p^2=0 \quad \to \quad s=\pm p,$$

2 there are two roots, corresponding to two solutions.

- Since $(s+1)^2-p^2 \neq 0$ for either of the roots, we have

1 to conclude that $a_1 \equiv 0$ in either case.

- The 3rd term in the eq. above gives the recurrence relation:

2 $$a_n = -\frac{a_{n-2}}{(n+s)^2-p^2}$$

- Since $a_1=0$, it is clear that only even coefficients

1 will survive. ~~The first 3 terms are~~

$\cancel{a_2 = -\frac{a_0}{\ }}$

- To work out the first 3 terms in the solutions, we relate a_2 and a_4 to a_0 using the recurrence relation:

2 $$a_2 = -\frac{a_0}{(s+2)^2-p^2} = \begin{cases} -\dfrac{a_0}{4(p+1)}, & \text{if } s=+p \\ -\dfrac{a_0}{4(-p+1)}, & \text{if } s=-p \end{cases}$$

$\cancel{a_4 = -\frac{a_2}{(4+s)^2-p^2}}$

2 $$a_4 = -\frac{a_2}{(4+s)^2-p^2} = \begin{cases} -\dfrac{a_2}{8(p+2)} = +\dfrac{a_0}{8\cdot 4(p+1)(p+2)}, & \text{if } s=+p \\ -\dfrac{a_2}{8(-p+2)} = +\dfrac{a_0}{8\cdot 4(-p+1)(-p+2)}, & \text{if } s=-p \end{cases}$$

so that the two solutions are:

1 $$y_1(x) = a_0\left[1 - \frac{x^2}{4(p+1)} + \frac{x^4}{8\cdot 4(p+1)(p+2)} - \dots\right]x^p$$

1 $$y_2(x) = a_0\left[1 - \frac{x^2}{4(-p+1)} + \frac{x^4}{8\cdot 4(-p+1)(-p+2)} - \dots\right]x^{-p}$$

(c) The general solution is

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

with arbitrary constants C_1, C_2

Note that a_0 is absorbed in both of them $\rightarrow$ set to $a_0 = 1$ in either of them.

2

(5) (a) We seek a solution of the wave eq.

1 $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \quad \underbrace{y(0,t) = y(L,t) = 0}_{\text{the boundary condition}}$$

1 in the form: $$y(x,t) = \varphi(x)\,\chi(t)$$

which gives:

$$\varphi''(x)\,\chi(t) = \frac{1}{c^2}\varphi(x)\,\chi''(t)$$

2 $$\frac{\varphi''(x)}{\varphi(x)} = \frac{1}{c^2}\frac{\chi''(t)}{\chi(t)} \equiv k \leftarrow \text{the separation constant}$$

Thus, we obtain two equations:

2 $$\underbrace{\varphi'' = k\varphi}_{\text{involves only } x} \quad \text{and} \quad \underbrace{\chi'' = kc^2\chi}_{\text{involves only } t}$$

(b) If $k = -p_n^2 \Rightarrow$ eq. involving x is

$$\varphi'' + p_n^2\varphi = 0$$

$\varphi_n(x) = \sin p_n x$ satisfies this equation since

2 $$\varphi_n''(x) = -p_n^2 \sin p_n x \equiv -p_n^2\varphi_n(x)$$

It also satisfies the boundary conditions:

$$\varphi_n(x=0) = 0$$

2 $$\varphi_n(x=L) = \sin(p_n L) = \sin\left(\frac{\pi}{L}n\cdot L\right) = \sin \pi n = 0$$

(c) Eq. involving t is

$$X_n''(t) + c^2 p_n X_n(t) = 0$$

$$X_n(t) = A_n \sin(c p_n t) + B_n \cos(c p_n t)$$

$$X_n'' = -c^2 p_n^2 \left[A_n \sin(c p_n t) + B_n \cos(c p_n t)\right]$$

3

$$= -c^2 p_n^2 X_n \quad , \quad \text{Q.E.D.}$$

(d) The general solution is a sum:

$$y(x,t) = \sum_{n=1}^{\infty} \psi_n(x)\, X_n(t)$$

1

$$= \sum_{n=1}^{\infty} \sin(p_n x)\left[A_n \sin(c p_n t) + B_n \cos(c p_n t)\right]$$

solutions:

2

$$y(x,t) = \sum_{n=1}^{\infty} \left[A_n \sin(c p_n t) + B_n \cos(c p_n t) \right] \sin(p_n x)$$

with A_n, B_n – arbitrary constants.

(e) • We use the given initial conditions to obtain the unknown constants A_n, B_n in the general solution.

2

• We have:

4

$$\begin{cases} y(x,0) = \sum_{n=1}^{\infty} B_n \sin(p_n x) \equiv \begin{cases} 0.3\frac{x}{L}, & 0 \le x \le \frac{L}{5} \\ 0.075\left(1-\frac{x}{L}\right), & \frac{L}{5} \le x \le L \end{cases} \\ y'_t(x,0) = \sum_{n=1}^{\infty} c p_n A_n \sin(p_n x) \equiv 0 \end{cases}$$

• Multiplying the 2nd equation on $\sin(p_m x)$ and integrating over x from 0 to L, we obtain $\boxed{A_m \equiv 0}$ since functions $\sin(p_n x)$ form an orthogonal set.

2

• Multiplying the 1st eq. on $\sin(p_m x)$, ~~and again~~ ~~using~~ integrating over x and applying the orthogonality relations, we obtain:

2

$$B_m \cdot \frac{L}{2} = \int_0^{L/5} 0.3\frac{x}{L} \sin p_m x \, dx + \int_{L/5}^{L} 0.075\left(1-\frac{x}{L}\right) \sin p_m x \, dx$$

Here we use integration by parts:

$$\int_0^{L/5} x \sin p_m x \, dx = \left[-\frac{1}{p_m} x \cos p_m x + \int \frac{\cos p_m x}{p_m} dx \right]_0^{L/5} =$$

$$= -\frac{L}{5p_m}\cos\frac{p_m L}{5} + \frac{1}{p_m^2}\sin\frac{p_m L}{5} = -\frac{L^2}{5\pi m}\cos\frac{\pi m}{5} +$$

2 $$+\frac{L^2}{\pi^2 m^2}\sin\frac{\pi m}{5},$$

$$\int_{L/5}^{L}\left(1-\frac{x}{L}\right)\sin p_m x\,dx = -\left(1-\frac{x}{L}\right)\frac{\cos p_m x}{p_m}\Bigg|_{L/5}^{L} +$$

$$+\int_{L/5}^{L}\left(-\frac{1}{L}\right)\frac{\cos p_m x}{p_m}\,dx = \frac{4}{5p_m}\cos\frac{p_m L}{5} - \frac{1}{Lp_m^2}\sin p_m x\Bigg|_{L/5}^{L}$$

2 $$= \frac{4L}{5\pi m}\cos\frac{\pi m}{5} + \frac{L}{\pi^2 m^2}\sin\frac{\pi m}{5}$$

- Collecting all terms, we get:

$$B_m\frac{L}{2} = \frac{0.3}{L}\left[\cancel{-\frac{L^2}{5\pi m}\cos\frac{\pi m}{5}} + \frac{L^2}{\pi^2 m^2}\sin\frac{\pi m}{5}\right] +$$

$$+0.075\left[\cancel{\frac{4L}{5\pi m}\cos\frac{\pi m}{5}} + \frac{L}{\pi^2 m^2}\sin\frac{\pi m}{5}\right]$$

$$B_m\frac{L}{2} = 0.375\,\frac{L}{\pi^2 m^2}\sin\frac{\pi m}{5}$$

1 $$\boxed{B_m = \frac{0.75}{\pi^2 m^2}\sin\frac{\pi m}{5}}$$

and finally, the partial solution reads:

1 $$y(x,t) = \sum_{n=1}^{\infty}\left[\frac{0.75}{\pi^2 n^2}\sin\frac{\pi n}{5}\right]\cos\left(\frac{c\pi n}{L}t\right)\cdot\sin\left(\frac{\pi n}{L}x\right)$$