King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the Authority of the Academic Board.

B.Sc. EXAMINATION

CM102D Mathematics for Science Students

Summer 2006

Time allowed: THREE Hours

Candidates should answer ALL questions from SECTION A,
and no more than TWO questions from SECTION B.
No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must NOT use a calculator for this paper.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer ALL parts of this section

1.1 The equation,

$$z^4 + 7z^3 + 16z^2 + 18z + 8 = 0,$$

where z may be complex, has one root at $z = -(1+i)$. Find all the roots of this equation.

[5 marks]

1.2. Evaluate the integral

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{4 - x^2}}.$$

[Hint: You may find a substitution of the form $x = a\sin\theta$ useful.]

[5 marks]

1.3 Solve the following equation for real x:

$$\sin x - 3\cos x = 3$$

[5 marks]

1.4 Evaluate the integral $\displaystyle\int_{y=1}^{2}\int_{x=y}^{2y} 8xy \,\mathrm{d}x\mathrm{d}y$.

[5 marks]

1.5 If $V = \ln(x^2 + y^2)$, prove that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$

[5 marks]

1.6 Solve the following simultaneous equations for z, which may be complex:

$$|z - 2i| = 2$$

$$\mathrm{Arg}(z) = \frac{\pi}{4}$$

Check your answer(s) by plotting the two equations on an Argand diagram.

[5 marks]

1.7 Find all the eigenvalues of the matrix $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$.

[5 marks]

1.8 Find a unit vector normal to the plane which passes through the following points: (1,1,1), (2,0,2), (1,1,2)

[5 marks]

SECTION B - Answer TWO questions

2. a) Show that

$$\int \frac{\mathrm{d}x}{\tan x} = \ln(|\sin x|) + c$$

where c is a constant of integration.

[4 marks]

b) Express $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$, where $z = e^{i\theta}$, and thereby show that

$$\sin^3\theta = \tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin 3\theta$$

[6 marks]

c) Use the method of integrating factors, and the results above, to solve the first order differential equation

$$\tan x \frac{\mathrm{d}y}{\mathrm{d}x} + y = \tan x \sin^2 x$$

[12 marks]

d) If $y = 0$ when $\tan x = 2$, find the particular solution.

[8 marks]

3. a) Show that $\sinh^{-1} x$ can be expressed as $\ln\left(x + \sqrt{x^2+1}\right)$.

[8 marks]

b) Differentiate both $\sinh^{-1} x$ and $\ln\left(x + \sqrt{x^2+1}\right)$ to show that, in both cases, the differential is $\left(x^2+1\right)^{-1/2}$.

[8 marks]

c) Expand $\sinh^{-1} x$ as a power series in x, up to and including terms in x^5.

[8 marks]

d) Differentiate this power series, term by term, and verify that it is the same series as that given by the binomial expansion of $\left(x^2+1\right)^{-1/2}$.

[6 marks]

4. a) Use the determinant method to solve the following simultaneous equations:

$$2x - 2y - z - 3 = 0$$
$$4x + 5y - 2z + 3 = 0$$
$$3x + 4y - 3z + 7 = 0$$

[8 marks]

b) If the third equation is replaced by:

$$8x + y - 4z - 3 = 0$$

show that the equations have no unique solution.

[4 marks]

c) Find the inverse of matrix $\boldsymbol{A} = \begin{pmatrix} 2 & -2 & -1 \\ 4 & 5 & -2 \\ 3 & 4 & -3 \end{pmatrix}$.

[12 marks]

d) Show how the equation $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{x}$ is the vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\boldsymbol{b}$ is given by $\begin{pmatrix} 3 \\ -3 \\ -7 \end{pmatrix}$, could be solved by matrix methods, and using the results of part (c), find the explicit solution for $\boldsymbol{x}$.

[6 marks]