Vibrations & Waves Homework Sheet 3 ANSWERS

1)i) Maximum power transfer at ω_0

***When:** $\omega = \omega_0 = \sqrt{s/m} = \sqrt{20{,}000/1} = 141.42 rad/s$

=> f = 22.51 Hz

ii) $$P_{av}(peak) = \frac{F_0^2}{2r} = \frac{(300)^2}{2 \times 100} = 450W$$

iii) Bandwidth:

$$\omega_2 - \omega_1 = \omega_0 / Q = r/m = 100/1 = 100 rad/s$$

$$\Delta f = f_2 - f_1 = 15.92 Hz$$

iv)

$$f_2 = f + \Delta f/2 = 22.51 + (15.92/2) = 30.47 Hz$$

$$f_1 = f - \Delta f/2 = 22.51 - (15.92/2) = 14.55 Hz$$

v) Angular frequency at maximum displacement:

$$\omega_r = \sqrt{\left(\omega_0^2 - \frac{r^2}{2m^2}\right)} = \sqrt{\left(20{,}000 - \frac{100^2}{2}\right)} = 122.474 rad/s$$

$$x_{max} = \frac{F_0}{\omega\sqrt{\left[r^2 + (\omega m - s/\omega)^2\right]}}$$

$$= \frac{300}{122.474\sqrt{\left[100^2 + (122.474 - 20{,}000/122.474)^2\right]}}$$

$$= 22.68 mm$$

Lightly damped decay:

$$x_{max}(t) = x_{max}(t=0)\exp\left(-\frac{r}{2m}t\right)$$

$$t = -\frac{2m}{r}\ln\left(\frac{x_{max}(t)}{x_{max}(t=0)}\right) = -\frac{2}{100}\ln\left(\frac{0.1}{22.68}\right) = 0.1085 s$$

No. of cycles: $0.1085 \times 122.474 / 2\pi = 2.12 cycles$

vi) Angular frequency at maximum power transfer:

$$\omega_0 = \sqrt{s/m} = \sqrt{20{,}000/1} = 141.42 rad/s$$

$$x_{\max} = \frac{F_0}{\omega\sqrt{\left[r^2 + (\omega m - s/\omega)^2\right]}}$$

$$= \frac{300}{141.42\sqrt{\left[100^2 + (141.42 - 20{,}000/141.42)^2\right]}}$$

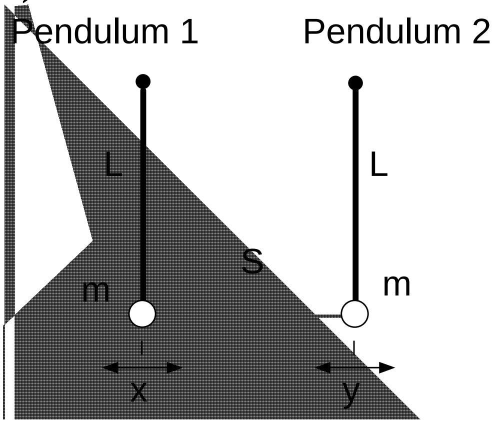
$$= 21.21mm$$

Lightly damped decay:

$$t = -\frac{2m}{r}\ln\left(\frac{x_{\max}(t)}{x_{\max}(t=0)}\right) = -\frac{2}{100}\ln\left(\frac{0.1}{21.21}\right) = 0.1071s$$

No. of cycles: $0.1085 \times 141.42 / 2\pi = 2.41 cycles$

2)



Pendulum 1:

$$m\frac{d^2x}{dt^2} = -\frac{mg}{L}x - S(x-y)$$

Pendulum 2:

$$m\frac{d^2y}{dt^2} = -\frac{mg}{L}y - S(y-x)$$

Define two *<u>Normal Co-ordinates</u>*

$$X = x + y$$
$$Y = x - y$$

Rearrange:

$$m\frac{d^2X}{dt^2} = -\frac{mg}{L}x - S(x-y) - \frac{mg}{L}y - S(y-x)$$

$$= -\frac{mg}{L}x - \frac{mg}{L}y = -\frac{mg}{L}X$$

and

$$m\frac{d^2Y}{dt^2} = -\frac{mg}{L}x - S(x-y) + \frac{mg}{L}y + S(y-x)$$

$$= -\frac{mg}{L}(x-y) - 2S(x-y) = -\left(\frac{mg}{L} + 2S\right)Y$$

Two *Normal Modes*:

Mode 1

$$X = A_1 \exp(j(\omega_1 t + \phi_1)) => A_1 \cos(\omega_1 t + \phi_1)$$

$$\omega_1 = \sqrt{g/L}$$

Mode 2

$$Y = A_2 \exp(j(\omega_2 t + \phi_2)) => A_2 \cos(\omega_2 t + \phi_2)$$

$$\omega_2 = \sqrt{(g/L + 2S/m)}$$

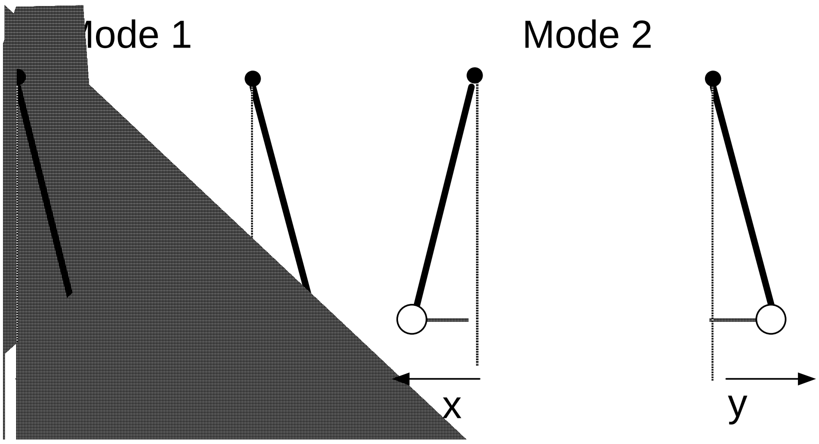
Therefore motion of individual pendulums:

$$x = \frac{1}{2}(X + Y)$$

$$= \frac{1}{2}[A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2)]$$

$$y = \frac{1}{2}(X - Y)$$

$$= \frac{1}{2}[A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2)]$$



3) Wave equation:

$$\frac{d^2\psi(x,t)}{dx^2} = \frac{1}{\mathrm{v}^2}\frac{d^2\psi(x,t)}{dt^2}$$

(i) $\psi(x,t) = 0.2\exp(j(628t - 0.01x + 3.14))$

$$\frac{\partial^2\psi(x,t)}{\partial t^2} = j^2 628^2\left(0.2\exp(j(628t - 0.01x + 3.14))\right)$$

$$= -394{,}384\psi(x,t)$$

$$\frac{\partial^2\psi(x,t)}{\partial x^2} = j^2(-0.01)^2\left(0.2\exp(j(628t - 0.01x + 3.14))\right)$$

$$= -0.0001\psi(x,t)$$

Therefore:

$$\frac{1}{-0.0001}\frac{\partial^2\psi(x,t)}{\partial x^2} = \psi(x,t) = \frac{1}{-394{,}384}\frac{\partial^2\psi(x,t)}{\partial t^2}$$

$$=> \frac{\partial^2\psi(x,t)}{\partial x^2} = 2.536\times 10^{-10}\frac{\partial^2\psi(x,t)}{\partial t^2}$$

So does obey wave equation where:

v = 628,000ms^{-1}

Also: T = 2π/628=0.01s

And: λ = 2π/0.01 = 628m

(ii) $\psi(x,t) = (0.1 - 0.15j)\exp(j(31.4t + 0.1x))$

$(0.1 - 0.15j)$ **=> complex amplitude (contains phase) => a constant**

$$\frac{\partial^2\psi(x,t)}{\partial t^2} = (j31.4)^2\psi(x,t) \quad \textbf{and} \quad \frac{\partial^2\psi(x,t)}{\partial x^2} = (j0.1)^2\psi(x,t)$$

Therefore:

$$\frac{1}{-0.01}\frac{\partial^2\psi(x,t)}{\partial x^2} = \psi(x,t) = \frac{1}{-985.96}\frac{\partial^2\psi(x,t)}{\partial t^2}$$

$$=> \frac{\partial^2\psi(x,t)}{\partial x^2} = 1.014\times 10^{-5}\frac{\partial^2\psi(x,t)}{\partial t^2}$$

So does obey wave equation where:

v = 314ms^{-1}

Also: T = 2π/31.4=0.2s

And: λ = 2π/0.1 = 62.8m

(iii) $\psi(x,t) = 2\cos(31.4t - 0.7x)$

$$\frac{\partial^2\psi(x,t)}{\partial t^2} = (31.4)^2\left(-2\cos(31.4t - 0.7x)\right) = -(31.4)^2\psi(x,t)$$

and $\frac{\partial^2\psi(x,t)}{\partial x^2} = (0.7)^2\left(-2\cos(31.4t - 0.7x)\right) = -(0.7)^2\psi(x,t)$

Therefore:

$$\frac{1}{-(0.7)^2}\frac{\partial^2\psi(x,t)}{\partial x^2} = \psi(x,t) = \frac{1}{-(31.4)^2}\frac{\partial^2\psi(x,t)}{\partial t^2}$$

$$=> \frac{\partial^2\psi(x,t)}{\partial x^2} = \left(\frac{0.7}{31.4}\right)^2\frac{\partial^2\psi(x,t)}{\partial t^2}$$

So does obey wave equation where:
v = 44.86ms^{-1}
Also: T = 2π/31.4=0.2s
And: λ = 2π/0.7 = 8.97m

(iv) $\psi(x,t) = 3\sin(628t - 0.7x + 1)$

$$\frac{\partial^2\psi(x,t)}{\partial t^2} = (628)^2\left(-3\sin(628t - 0.7x + 1)\right) = -(628)^2\psi(x,t)$$

and $\frac{\partial^2\psi(x,t)}{\partial x^2} = (-0.7)^2\left(-3\sin(628t - 0.7x + 1)\right) = -(0.7)^2\psi(x,t)$

Therefore:

$$\frac{1}{-(0.7)^2}\frac{\partial^2\psi(x,t)}{\partial x^2} = \psi(x,t) = \frac{1}{-(628)^2}\frac{\partial^2\psi(x,t)}{\partial t^2}$$

$$=> \frac{\partial^2\psi(x,t)}{\partial x^2} = \left(\frac{0.7}{628}\right)^2\frac{\partial^2\psi(x,t)}{\partial t^2}$$

So does obey wave equation where:
v = 897ms^{-1}
Also: T = 2π/628=0.01s
And: λ = 2π/0.7 = 8.97m

(v) $\psi(x,t) = 5\cos(100t - 30x)\sin(100t - 30x)$

$$\frac{\partial\psi(x,t)}{\partial t} = 100\left[-5\sin(100t - 30x)\sin(100t - 30x)\right.$$
$$\left. +5\cos(100t - 30x)\cos(100t - 30x)\right]$$

can continue to differentiate by parts or use a good trig. substitution such as: $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

$$=> \frac{\partial\psi(x,t)}{\partial t} = 100\left[5\cos(2(100t - 30x))\right]$$

$$=> \frac{\partial^2\psi(x,t)}{\partial t^2} = 2(100)^2\left[-5\sin(2(100t - 30x))\right]$$

and: $$\frac{\partial\psi(x,t)}{\partial x} = -30\left[-5\sin(100t - 30x)\sin(100t - 30x)\right.$$
$$\left. +5\cos(100t - 30x)\cos(100t - 30x)\right]$$

use same trig substitution:

$$\frac{\partial\psi(x,t)}{\partial x} = -30\left[5\cos(2(100t - 30x))\right]$$

$$=> \frac{\partial^2\psi(x,t)}{\partial x^2} = 2(-30)^2\left[-5\sin(2(100t - 30x))\right]$$

Therefore:

$$\frac{1}{(-30)^2}\frac{\partial^2\psi(x,t)}{\partial x^2} = 2\left[-5\sin(2(100t - 30x))\right] = \frac{1}{(100)^2}\frac{\partial^2\psi(x,t)}{\partial t^2}$$

$$=> \frac{\partial^2\psi(x,t)}{\partial x^2} = \left(\frac{30}{100}\right)^2\frac{\partial^2\psi(x,t)}{\partial t^2}$$

So does obey wave equation where:

v = 3.33ms^{-1}

Also: T = 2π/100=0.063s

And: λ = 2π/30 = 0.209m

4) Water: density ρ, bulk modulus B.

$$F = B\frac{\partial^2 X(x,t)}{\partial x^2}a\Delta x$$

Newton II:

$$F = ma = m\frac{d^2X(x,t)}{dt^2} = \rho a\Delta x\frac{d^2X(x,t)}{dt^2}$$

=> $$\rho\frac{d^2X(x,t)}{dt^2} = B\frac{d^2X(x,t)}{dx^2}$$

=> $$\frac{\rho}{B} = \frac{1}{\mathrm{v}^2}$$

ρ = 10^3 kg/m^3, bulk modulus B = 2.18x10^9 N/m^2

=> Velocity of the sound waves in water:

$$\mathrm{v} = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.18\times 10^9}{10^3}} = 1476\, m/s$$

Derive an equation for the average power contained in the sound waves:

Sound wave:

$$X(x,t) = X_0 \exp(j(\omega t - kx + \phi))$$

real part:

$$X(x,t) = X_0 \cos(\omega t - kx + \phi)$$

velocity of water molecule:

$$v_X(x,t) = -\omega X_0 \sin(\omega t - kx + \phi)$$

Kinetic energy of mass element $\rho \Delta x a$

$$KE(x,t) = \frac{1}{2}\rho \Delta x a(\omega X_0 \sin(\omega t - kx + \phi))^2$$

Therefore, total energy per wavelength λ**:**

$$TE = \frac{1}{2}\rho \lambda a \omega^2 X_0^2$$

Total energy arriving at any point on x **per period** T**:**

$$\frac{TE}{T} = \frac{1}{2}\rho \frac{\lambda}{T} a \omega^2 X_0^2 = \frac{1}{2}\rho v a \omega^2 X_0^2 = \frac{1}{2}\rho v^2 a \omega k X_0^2$$

$$= \frac{1}{2}\rho \frac{B}{\rho} a \omega k X_0^2 = \frac{1}{2} B a \omega k X_0^2$$

Time average power:

$$P_{av} = \frac{1}{2} B a \omega k X_0^2$$

Intensity:

$$I = \frac{P_{av}}{a} = \frac{1}{2} B \omega k X_0^2$$

The sonar emits a pulse of average power 10 W at a frequency of 1 kHz. Assume that the sonar emitter is a sphere of radius 10 cm. What is the maximum displacement of a water molecule at the sonar emitter?

$$X_0 = \sqrt{\frac{2P_{av}}{aB\omega k}}$$

$$a = 4\pi r^2 = 4\pi(0.1)^2 = 0.1257m^2$$

$$\omega = 2\pi\nu = 6{,}280 rad/s$$

$$k = \omega/\mathrm{v} = 6{,}280/1476 = 4.255 rad/m$$

$$=> X_0 = \sqrt{\frac{2\times 10}{0.1257\times 2.18\times 10^9 \times 6280\times 4.255}} = 1.65\times 10^{-6} m$$

=> 1.65 μm

What is the intensity of the sound waves at a distance of 100m from the sonar emitter?

$$I = P_{av}/a = P_{av}/(4\pi r^2) = 10/(4\pi(100)^2) = 7.96\times 10^{-5} W/m^2$$