Structure of Matter

Problem Sheet 2 **Answers**

1. Total probability = 1, so $\int_{-\infty}^{\infty} f(v_x)\, \mathrm{d}v_x = 1$

   $$\rightarrow \int_{-\infty}^{\infty} A\mathrm{e}^{-\alpha v_x^2}\, \mathrm{d}v_x = A \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha v_x^2}\, \mathrm{d}v_x = A\, \left(\frac{\pi}{\alpha}\right)^{1/2} = 1$$

   $$\rightarrow A = \left(\frac{\alpha}{\pi}\right)^{1/2} = \left(\frac{m}{2\pi k_B T}\right)^{1/2}$$

2. Most probable speed is a turning point (maximum) of the function $f(v) = A^3\, 4\pi v^2\, \mathrm{e}^{-\alpha v^2}$

   (A and α have the same meanings as in Q1 here and in the following questions)

   i.e. $\dfrac{\mathrm{d}f(v)}{\mathrm{d}v} = 4\pi A^3 \left(2v\, \mathrm{e}^{-\alpha v^2} - 2\alpha v\, v^2\, \mathrm{e}^{-\alpha v^2}\right) = 0$

   $\rightarrow v\, \mathrm{e}^{-\alpha v^2}\, (1 - \alpha v^2) = 0$

   which gives the only non-trivial soln. as $v_{mp} = \left(\dfrac{1}{\alpha}\right)^{1/2} = \left(\dfrac{2k_B T}{m}\right)^{1/2}$

3. $\langle v \rangle = 4\pi A^3 \int_0^{\infty} v^3 \mathrm{e}^{-\alpha v^2} \mathrm{d}v = 4\pi A^3 \dfrac{1}{2\alpha^2}$

   where we have used one of the standard integrals. Using (as from Q1) $A = \left(\dfrac{\alpha}{\pi}\right)^{1/2}$,

   $$\langle v \rangle = 4\pi \left(\frac{\alpha}{\pi}\right)^{3/2} \frac{1}{2\alpha^2} = 2\left(\frac{1}{\pi\alpha}\right)^{1/2} = \left(\frac{8k_B T}{\pi m}\right)^{1/2}$$

4. $\langle v^2 \rangle = 4\pi A^3 \int_0^{\infty} v^4 \mathrm{e}^{-\alpha v^2} \mathrm{d}v = 4\pi A^3 \times \dfrac{3}{8}\left(\dfrac{\pi}{\alpha^5}\right)^{1/2}$

   again making use of a standard integrals. With $A = \left(\dfrac{\alpha}{\pi}\right)^{1/2}$,

   $$\langle v^2 \rangle = 4\pi \left(\frac{\alpha}{\pi}\right)^{3/2} \frac{3}{8}\left(\frac{\pi}{\alpha^5}\right)^{1/2} = \frac{3}{2}\left(\frac{1}{\alpha}\right) = \left(\frac{3k_B T}{m}\right)$$

   The mean kinetic energy is then $\bar{u} = \langle \frac{1}{2} m v^2 \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} k_B T$

5. $v_{mp} = \left(\dfrac{2k_B T}{m}\right)^{1/2} = \sqrt{2}\, \left(\dfrac{1.38 \times 10^{-23} \cdot 293}{32 \cdot 1.66 \times 10^{-27}}\right)^{1/2} = 390\ \mathrm{ms}^{-1}$

$$\langle v\rangle = \left(\frac{8k_BT}{\pi m}\right)^{1/2} = \left(\frac{8}{\pi}\right)^{1/2} \left(\frac{1.38\times10^{-23}\cdot 293}{32\cdot 1.66\times10^{-27}}\right)^{1/2} = 440\ \text{ms}^{-1}$$

$$\langle v^2\rangle^{1/2} = \left(\frac{3k_BT}{m}\right)^{1/2} = \sqrt{3}\left(\frac{1.38\times10^{-23}\cdot 293}{32\cdot 1.66\times10^{-27}}\right)^{1/2} = 478\ \text{ms}^{-1}$$

6. (a) $$\langle v_D\rangle = \left(\frac{8k_BT}{\pi m_D}\right)^{1/2} = \left(\frac{8}{\pi}\right)^{1/2} \left(\frac{1.38\times10^{-23}\cdot 10^8}{3.34\times10^{-27}}\right)^{1/2} = 1.03\times10^{6}\ \text{ms}^{-1}$$

$$\langle v_T\rangle = \left(\frac{8k_BT}{\pi m_T}\right)^{1/2} = \left(\frac{8}{\pi}\right)^{1/2} \left(\frac{1.38\times10^{-23}\cdot 10^8}{5.01\times10^{-27}}\right)^{1/2} = 8.37\times10^{5}\ \text{ms}^{-1}$$

$$\langle v_e\rangle = \left(\frac{8k_BT}{\pi m_e}\right)^{1/2} = \left(\frac{8}{\pi}\right)^{1/2} \left(\frac{1.38\times10^{-23}\cdot 10^8}{9.11\times10^{-31}}\right)^{1/2} = 6.21\times10^{7}\ \text{ms}^{-1}$$

(b) for all species, $\bar{u} = \frac{3}{2}k_BT$ (independent of m)

$$\bar{u} = \tfrac{3}{2}k_BT = \tfrac{3}{2}\cdot 1.38\times10^{-23}\cdot 10^8 = 2.07\times10^{15}\text{J} = 13{,}000\ \text{eV} = 13\ \text{keV}$$

7. $$\langle v_x\rangle = \int_{-\infty}^{\infty} v_x f(v_x)\,\mathrm{d}v_x = \int_{-\infty}^{\infty} v_x\, A\mathrm{e}^{-\alpha v_x^2}\,\mathrm{d}v_x \text{ (directly integratable)}$$

$$= \left[\frac{A}{2}\mathrm{e}^{-\alpha v_x^2}\right]_{-\infty}^{+\infty} = \frac{A}{2}\left[\mathrm{e}^{-\infty} - \mathrm{e}^{-\infty}\right] = 0$$

8. $$\langle z\rangle = \int_0^\infty p(z)\mathrm{d}z$$

$$= \textstyle\int_0^\infty \frac{z}{\lambda}\mathrm{e}^{-z/\lambda}\mathrm{d}z = \left(\left[-z\mathrm{e}^{-z/\lambda}\right]_0^\infty + \int_0^\infty \mathrm{e}^{-z/\lambda}\mathrm{d}z\right) \text{ (integrating by parts)}$$

$$= 0 + \left[-\lambda\,\mathrm{e}^{-z/\lambda}\right]_0^\infty$$

$$= 0 - (-\lambda) = \lambda$$

So $<\text{p.e.}>= \langle mgz\rangle = mg\langle z\rangle = mg\lambda = mg\dfrac{k_BT}{mg} = k_BT$