UNIVERSITY OF LONDON

BSc/MSci EXAMINATION May 2005

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

PHYSICS I Comprehensive Paper

For Third- and Fourth-Year Physics Students

Friday 20th May 2005: 10.00 to 13.00

Answer FIVE Questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FIVE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FIVE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

1. Whilst orbiting the Earth an astronaut leaves a spacecraft and undertakes a space walk when unfortunately her jet pack fails. Her only connection to the spacecraft is a communication wire of length $L = 100\,\text{m}$.

You may assume the height of the spacecraft's orbit is negligible compared to the Earth's radius of 6400 km. Also assume that the astronaut and spacecraft remain on a ray projecting from the Earth's centre with the astronaut further away from the Earth than the spacecraft.

(i) Explain what is meant by a *centripetal force*, and comment on this with respect to the situation of a spacecraft orbiting the Earth. [2 marks]

(ii) Explain why an expression for the forces acting on the spacecraft is given by

$$\frac{GM_SM_E}{R^2} - T = M_S\Omega^2 R$$

where G is the gravitational constant, T the wire tension, M_S is the mass of the spacecraft, M_E is the mass of the Earth, Ω is the angular velocity of the spacecraft, and R is the distance from the Earth's centre to the spacecraft. [4 marks]

(iii) Neglecting the gravitational attraction between the astronaut and spacecraft, derive a similar expression to that of (ii) for the forces on the astronaut of mass m. [3 marks]

(iv) Derive an expression for the tension T in the communication wire in terms of m, M_S, M_E, G, R, and R_2, where $R_2 = R + L$. [4 marks]

(v) Simplify your expression for T using assumptions given above, and taking $L \ll R$ and $m \ll M_S$, to show that T may be given by:

$$T = \frac{3mgL}{R}$$

where g is the acceleration due to gravity, L is the length of the communication wire. [4 marks]

(vi) Using the expression for T from (v), make a reasonable estimate for the value for T. Comment on whether you think the wire would be enough to prevent the astronaut drifting away off into space?

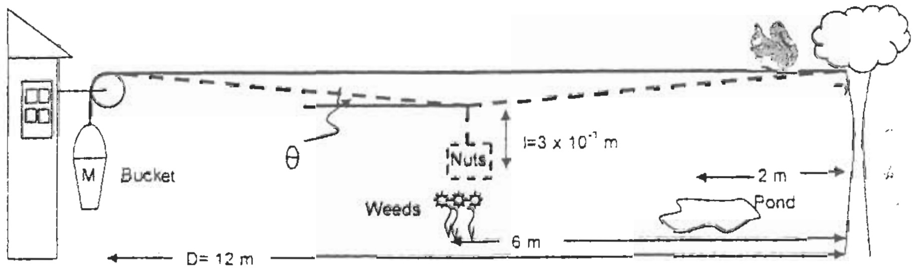
[take acceleration due to gravity for Earth, $g = 9.8\,\text{ms}^{-2}$]. [3 marks]

[TOTAL 20 marks]

2. (i) A string, of mass per unit length ρ and under tension T, can support transverse waves whose lateral displacements, $y(x, t)$ satisfy the wave equation:-

$$T\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2}.$$

Show that these waves propagate with a velocity $v = (T/\rho)^{1/2}$. [5 marks]



(ii) A squirrel has learnt to walk along a garden clothesline, but only at a fixed speed of $2 \times 10^{-1}\,\text{m}\,\text{sec}^{-1}$, grasping the clothesline every 3×10^{-2} m. His movements excite standing waves in the clothesline which, if they are resonant, become large enough to cause him to loose his footing and fall. The line, of length D, is tensioned by a mass M of water in the bucket, where M is much larger than the masses of the squirrel, line and bucket combined.

(a) Calculate the angular frequency, ω_{sq}, of the periodic driving force generated by the squirrel. [2 marks]

(b) Derive an expression for the resonant frequencies of the clothesline, in terms of ρ, D and M. [3 marks]

(iii) If the clothesline has $\rho = 10^{-2}\,\text{Kg}\,\text{m}^{-1}$, calculate the mass, M, of water the bucket should contain to maximise the chances of the squirrel falling off into

(a) the weeds. [2 marks]

(b) the pond. [2 marks]

(iv) Now a container of nuts, of mass m, is suspended from the middle of the line on a string of length l. So much water has to be added to the bucket to lift the nuts off the ground that the previous wave-like resonances become too high in frequency to bother the squirrel. Now his problems come from pendulum oscillations as the nuts swing around. Ignoring the masses of the lines, derive expressions, in terms of D, l and θ, for the angular frequencies with which the nuts will swing

(a) in the plane of the diagram. [2 marks]

(b) normal to the plane of the diagram. [2 marks]

(v) Which of these two new resonances will be most troublesome for the squirrel? *Explain your reasoning carefully.* [2 marks]

[data; $g = 9.81\,\text{m}\,\text{sec}^{-2}$]. [TOTAL 20 marks]

3. A magnet is released from rest at the top of a vertical metal tube of length L. The magnetic moment, $\boldsymbol{\mu}$, points vertically down, and the field lines penetrate the conducting walls of the tube. The acceleration due to gravity is $\mathbf{g}$.

(i) Write down Faraday's Law in integral form. [2 marks]

(ii) Sketch the magnetic field. As the magnet falls through the tube, indicate clearly the sense of the induced current. [3 marks]

(iii) Give Maxwell's equation for the induced magnetic field $\boldsymbol{B}$. What is the stored magnetic energy? Hence give an expression for the retarding force. [5 marks]

(iv) Let us assume that the motion is in fact described by the equation:

$$\ddot{z} = g - \alpha \dot{z} \,,$$

where $g = |\mathbf{g}|$.
What physical parameters in the experiment determine α? [4 marks]

(v) Show that when t is small $\dot{z} \approx gt$ and when t is large $\dot{z} \approx g/\alpha$. [4 marks]

(vi) Where is the effective 'frictional' energy dissipated? [2 marks]

[TOTAL 20 marks]

4. A particular final state, produced in electron-positron annihilation at high energy, contains an unstable neutral particle, K. K decays to two oppositely charged particles of equal mass.

(i) Define what is meant by an inertial frame and state the principle of relativity.

[2 marks]

(ii) What is meant by the rest mass of a particle? Write down an expression for the rest mass of a particle in terms of its energy and momentum. Show that the rest mass is invariant under the Lorentz Transformation (the Lorentz transformation equations are reproduced below).

[6 marks]

(iii) The energy and momentum of the two charged particles produced when K decays, which were measured precisely, are given in the following table:

Decay products				
		Components of Momentum (GeV/c)		
	Energy (GeV)	x component	y component	z component
Positive particle	4.121	0.799	0.570	4.000
Negative particle	6.085	0.900	0.445	6.000

What is the rest mass of K? What is the magnitude (in natural units, i.e. in units of c) and direction of the velocity of the neutral particle in the laboratory frame?

[6 marks]

(iv) If K had decayed at rest, what would have been the energy and the momentum of each of the decay products. (Note that you are not required to calculate the momentum components of the decay products in the rest frame of the neutral particle).

[2 marks]

(v) The point at which the positive and negative particles are observed to have been created is $(x, y, z) = (0.03, 0.06, 0.15)$ m. Are the measurements consistent with the hypothesis that K originated at the point at which the electron-positron annihilation took place? You may assume that the resolution of the experiment is so good that measurement uncertainties may be neglected.

[4 marks]

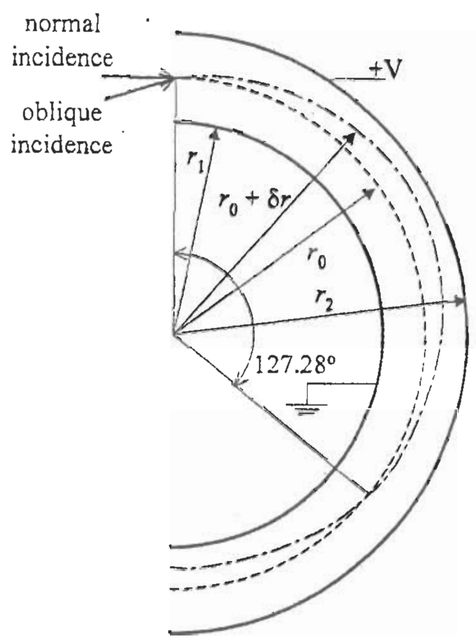
[TOTAL 20 marks]

[The Lorentz transformation of energy and momentum between two inertial frames S and S' may be written

$$cp_x = \gamma(cp'_x + \beta E'), \quad cp_y = cp'_y, \quad cp_z = cp'_z, \quad E = \gamma(E' + \beta cp'_x)$$

where the velocity with which S' moves relative to S is u in the $+x$ direction, $\beta = u/c$ (c is the speed of light) and $\gamma = \left(1 - \beta^2\right)^{-\frac{1}{2}}$.]

5. A semi-cylindrical capacitor consists of two infinite co-axial, semicircular conductor plates as shown in cross section in the Figure below. The inner plate is grounded and has a radius r_1, the outer plate, at a potential V, has a radius r_2. We also define the central radius between the plates as $r_0 = r_1 + (r_2 - r_1)/2$.



(i) The electric field E between the plates of the capacitor (neglecting edge effects) has only a radial component E_r.

(a) Define an appropriate closed surface to apply Gauss' law and derive the relationship between E_r and the electric charge q per unit length (defined in units of $C.m^{-1}$) on the capacitor. [2 marks]

(b) State the integral relationship between the electric field between the plates and the potential V. [2 marks]

(c) Use the results from (a) and (b) to calculate the electric field E_r between the plates of the capacitor. [2 marks]

(d) Show that for $r = r_0$ it can be approximated as

$$E_{r0} \approx \frac{V}{\Delta r}$$

where $\Delta r = r_2 - r_1 = \Delta r \ll r_0$. [2 marks]

(ii) A positively charged particle of mass M, charge Q is incident normally at velocity v_0 at the centre point r_0 between the plates as shown in the Figure opposite.

(a) Show that if the particle moves between the plates along the circular path at constant radius r_0 the ratio of its kinetic energy W to its electric charge Q satisfies

$$\frac{W}{Q} \approx \frac{V}{2} \frac{r_0}{\Delta r}.$$

[2 marks]

(b) We assume that a particle is incident with a small angular deviation from the normal at the centre point ($r = r_0$) between the plates (oblique incidence). We denote by ω the particle angular velocity in the electric field between the plates and by ω_0 the angular velocity of the particle that moves along the centre line of the capacitor (where $r = r_0$). Derive the equation of motion of the particle.

[2 marks]

(c) Assuming that the trajectory of the obliquely incident particle (starting at $r = r_0$) remains close to the circular motion between the plates, its position in the radial direction along its trajectory can be written as $r_0 + \delta r$ where $\delta r \ll r_0$. Using the full expression found for the electric field in (i) (c), show that the electric field E_r along the particle trajectory as a function of δr can be approximated by

$$E_r = E_{r0} \left(1 - \frac{\delta r}{r_0}\right).$$

[2 marks]

(d) The normally and obliquely incident particles can be assumed to have the same angular momentum. Using the equation that describes the motion of the normally incident particle, show that the obliquely incident particle's motion is described, using only first order terms in δr, by the equation

$$\frac{d^2(\delta r)}{dt^2} + 2\omega_0^2(\delta r) \approx 0.$$

[4 marks]

(e) By recognising the nature of this equation, show that obliquely incident particles at $r = r_0$ will be focussed again at $r = r_0$ after a nearly circular trajectory of angular extent 127.28°.

[2 marks]

[TOTAL 20 marks]

6. (i) Write down the 1-dimensional, time-independent Schrödinger Equation for a particle of mass m, total energy E, moving within a potential energy $U(x)$. [3 marks]

(ii) A particle is contained within a 1-dimensional, infinitely deep, square potential well. Write down the boundary condition for the particle's wave function at the walls. Briefly explain the physical reason for this condition. [3 marks]

(iii) The well lies between $x = 0$ and $x = +L$. The potential energy within the well is zero. Solutions of the Schrödinger Equation for the particle in the well are $\psi = A \sin K_n x$ where n is a positive integer. Show that

$$k_n = \frac{\sqrt{2mE_n}}{\hbar}$$

where E_n is the particle's energy. Hence show that

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}.$$

Sketch, with appropriate labels, the particle's eigenfunctions for the first 3 energy levels. [6 marks]

(iv) A closed cube of side L contains a Fermion gas. By extension of part (iii) to three dimensions, it may be shown that $E_n = \left(n_x^2 + n_y^2 + n_z^2\right) \pi^2 \hbar^2 / 2mL^2$, where n_x, n_y, n_z is a set of positive integers which identify the state. The total number of energy states with $E_n \leq E_F$ is $N = \pi n_F^3/3$, where E_F is the Fermi energy and $n_F^2 = n_{Fx}^2 + n_{Fy}^2 + n_{Fz}^2$.
Show that

$$N = \frac{(2mE_F)^{\frac{3}{2}} V}{3\pi^2 \hbar^3}$$

where V is the volume of the cube.
If the temperature of the gas is much less than E_F/k_B, explain briefly the physical significance of the Fermi Energy in this situation. [3 marks]

(v) A neutron star is a gravitationally bound sphere consisting mainly of neutrons. The gravitational binding energy is so strong that the neutrons can be regarded as lying within an infinite potential well. If the highest energy neutrons approach relativistic energies (i.e. kinetic energy $\approx$ rest mass energy) then the neutron star will collapse into a black hole. This occurs when the neutron star mass exceeds about twice the mass of the sun. Estimate the radius in kilometres of the neutron star at the moment it starts to collapse. (You may assume $T = 0\,\mathrm{K}$ within the star.)

[Rest mass of neutron $= 1.67 \times 10^{-27}$ kg
Mass of sun $= 2 \times 10^{30}$ kg] [5 marks]

[TOTAL 20 marks]

7. Consider a diatomic gas. According to quantum mechanics, the molecules possess rotational energy levels

$$E_\ell = \frac{\hbar^2}{2I}\ell(\ell+1), \qquad \ell = 0, 1, 2, \ldots$$

where I is a constant. The energy level E_ℓ is degenerate with quantum number $m_\ell = 0, \pm 1, \ldots, \pm \ell$.

(i) Show that the partition function of the rotational motion

$$Z_{\text{rot}} = \sum_{\ell=0}^{\infty}(2\ell+1)\exp\left(-\frac{\hbar^2}{2Ik_{\text{B}}T}\ell(\ell+1)\right).$$

[3 marks]

(ii) In the following we consider the low temperature limit $k_{\text{B}}T \ll \hbar^2/(2I)$.

(a) Argue why only the lowest levels will be relevant in the partition function.

[3 marks]

(b) Hence show that

$$Z_{\text{rot}} \approx 1 + 3\exp\left(-\hbar^2/(Ik_{\text{B}}T)\right).$$

[2 marks]

(c) Show that the average rotational energy

$$\langle E_{\text{rot}} \rangle \approx 3\frac{\hbar^2}{I}\exp\left(-\hbar^2/(Ik_{\text{B}}T)\right).$$

[2 marks]

(iii) In the following we consider the high temperature limit $k_{\text{B}}T \gg \hbar^2/(2I)$.

(a) Define $x = \ell(\ell+1)$. Argue why

$$Z_{\text{rot}} \approx \int_0^\infty dx \exp\left(-\frac{\hbar^2}{2Ik_{\text{B}}T}x\right).$$

[3 marks]

(b) Find the partition function in the high temperature limit. [2 marks]

(c) Show that the average rotational energy

$$\langle E_{\text{rot}} \rangle \approx k_{\text{B}}T.$$

[2 marks]

(iv) (a) Explain which of the above results for the average rotational energy is classical.

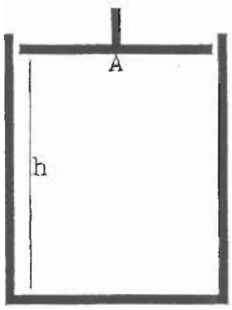
[1 mark]

(b) Hence explain this result for the average rotational energy in classical terms.

[2 marks]

[You may use, without proof, that $\int_0^\infty u\exp(-u)du = 1$.] [TOTAL 20 marks]

8. An office chair can be adjusted in height with a spring mechanism, and a gas cylinder is used to provide comfort because it cushions sudden motions.



We are taking a closer look at the gas cylinder. We will approximate the shape of the gas cylinder by a simple piston, with surface A and internal height h. Without any load additional to the weight of the chair the height of the piston is h_0, which we can presume known.

(i) Assume the gas inside the piston behaves as a perfect gas. Someone with a mass M is sitting on the chair, feet not touching the ground. Demonstrate that the equilibrium height h_e of the piston satisfies :

$$h_e = \frac{1}{\frac{Mg}{Nk_BT} + \frac{1}{h_0}}$$

in which T is the (fixed) gas temperature and g is the acceleration of gravity. N is the number of gas molecules in the cylinder and k_B is Boltzmann's constant.

[5 marks]

(ii) (a) If there is a small perturbation δh of the height h with respect to the equilibrium value h_e (while the person is sitting on it), show that, if we assume that the temperature T is constant, the restoring force $F = -Nk_BT\delta h/h_e^2$, and hence write down the equation of motion.

[5 marks]

(b) Someone pushes the person sitting on the chair down and then lets go. Demonstrate that the solution of the equation of motion is a harmonic oscillation with period $\Pi = 2\pi h_e\sqrt{\frac{M}{Nk_BT}}$. (The mass of the chair is included in M). [3 marks]

(iii) A perfect gas undergoing adiabatic change obeys the equation $pV^\gamma = p_0V_0^\gamma$ where $\gamma = 5/3$ for a monatomic gas and $\gamma = 7/3$ for a diatomic gas.

(a) Explain what is meant by an adiabatic change. Suggest why this might be a more appropriate assumption for the gas than the isothermal assumption used in part (ii)(b).

[2 marks]

(b) Calculate the restoring force for part (ii) for an adiabatic change. What is now the period of the oscillation?

[4 marks]

(c) How could you tell whether the piston was filled with air or with Argon ?

[1 mark]

[TOTAL 20 marks]

9. (i) Heat is lost from a hot water tank, of total heat capacity C and temperature T, through an insulating jacket of thickness l, surface area A and thermal conductivity K, to the environment at temperature T_a. Based on the definition of thermal conductivity explain why the rate of change of temperature of the water can be given by:

$$\frac{dT(t)}{dt} = -\frac{KA}{Cl}(T(t) - T_a) .$$

[2 marks]

(ii) By substitution for $(T(t) - T_a)$ integrate this equation to find an expression for the water temperature as a function of time as it cools from its initial value T_0.

[2 marks]

(iii) At $t = t_1$, when the water temperature is T_1, a water heater of power W is switched on. Show that, for $t \geq t_1$:

$$T(t) = T_a + \frac{\tau W}{C} - \left(T_A + \frac{\tau W}{C} - T_1\right) \exp\left(-(t - t_1)/\tau\right)$$

where $\tau = Cl/KA$.

[3 marks]

(iv) A thermostat is fitted to the tank such that the heater is switched off when the water temperature reaches T_{req} and switched on when it drops to T_{min}. Make a carefully labelled sketch of the behaviour of $T(t)$ over two heating/cooling cycles. [It can be assumed that T_{req} and T_{min} are physically attainable].

[4 marks]

(v) If, at $t = 0$, $T_0 = T_{req}$ show that the time at which the heater is first switched on is:

$$t_1 = \tau \ln \frac{(T_{req} - T_a)}{(T_{min} - T_a)}$$

and find an expression for t_2, the time at which it switches off again.

[5 marks]

(vi) Evaluate t_1 and t_2 with $W = 3\,\text{kW}$, $C = 10^6\,\text{JK}^{-1}$, $l = 2\,\text{cm}$, $A = 2\,\text{m}^2$, $K = 0.04\,\text{W}\,\text{m}^{-1}\,\text{K}^{-1}$ for $T_{req} = 60\,°\text{C}$, $T_a = 10\,°\text{C}$ if

(a) $T_{min} = 58\,°\text{C}$ and

(b) $T_{min} = 18\,°\text{C}$.

[2 marks]

(vii) After your morning shower you do not need hot water again until the evening. What factors will determine whether it is more energy efficient to leave the water heater (with thermostat) switched on or off during the day?

[Algebra/maths are not required for this part]

[2 marks]

[TOTAL 20 marks]

10. Write an essay about ONE of the following topics.

(i) Should maintenance of the Hubble Space Telescope continue?

(ii) As the number of students studying A-level science is declining should UK universities be closing science departments?

(iii) "God does not play dice" (Albert Einstein).

(iv) The potential construction of the European Spallation Source.

(v) The representation of scientists in the media.

(vi) The impact of climate change on Europe.

[20 marks]

End