M2AA2 - Multivariable Calculus. Problem Sheet 9. Solutions.
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1. (a)
$$\delta_{ij}\frac{\partial x_i}{\partial x_j} = \frac{\partial x_i}{\partial x_i} = 3$$
(b)
$$\delta_{ij}\delta_{ik}x_jx_k = \delta_{jk}x_jx_k = x_jx_j = x_1^2 + x_2^2 + x_3^2 = r^2$$
(c)
$$\delta_{ij}\frac{\partial^2\phi}{\partial x_i\partial x_j} = \frac{\partial^2\phi}{\partial x_i\partial x_i} = \nabla^2\phi$$
(d)
$$\delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ik}\delta_{ki} = \delta_{ii} = 3$$

2. We have from the definition of angular momentum and the fact that $\boldsymbol{v} = \boldsymbol{\omega}\times\boldsymbol{x}$ that
$$\begin{aligned} h_i = m\epsilon_{ijk}x_jv_k = m\epsilon_{ijk}x_j\epsilon_{kpq}\omega_px_q = m\epsilon_{kij}\epsilon_{kpq}x_j\omega_px_q = m(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})x_j\omega_px_q \\ = m(x_q\omega_ix_q - x_p\omega_px_i) = m(x_kx_k\delta_{ij}\omega_j - x_jx_j\omega_j) \equiv I_{ij}\omega_j, \end{aligned}$$
where I_{ij} is the *inertia tensor.*

3.
$$\begin{aligned} [\boldsymbol{u}\times(\nabla\times\boldsymbol{u})]_i = \epsilon_{ijk}u_j\epsilon_{kpq}\frac{\partial u_q}{\partial x_p} = (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})u_j\frac{\partial u_q}{\partial x_p} \\ = u_j\frac{\partial u_j}{\partial x_i} - u_j\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\frac{\partial u_j^2}{\partial x_i} - u_j\frac{\partial u_i}{\partial x_j}, \end{aligned}$$
as required.

4. (a)
$$\phi' = u'v' = a_{ij}u_ja_{jk}v_k = \delta_{jk}u_jv_k = u_jv_j = \phi,$$
i.e. a scalar, hence a tensor of rank zero.
$$T_i' = u_k'u_k'v_i' = a_{kr}u_ra_{ks}u_sa_{it}v_t = \delta_{rs}u_ru_sa_{it}v_t = a_{it}(u_ru_rv_t) = a_{it}T_t,$$
hence a tensor of rank one.

(b) Transform T_{ij} to T_{ij}' as follows
$$T_{ij}' = \frac{\partial A_i'}{\partial x_j'} = \left(\frac{\partial A_i'}{\partial x_r}\frac{\partial x_r}{\partial x_j'}\right) = a_{is}\frac{\partial A_s}{\partial x_r}\frac{\partial x_r}{\partial x_j'}.$$
Since $x_r = a_{tr}x_t'$ we have $\frac{\partial x_r}{\partial x_j'} = a_{tr}\frac{\partial x_t'}{\partial x_j'} = a_{tr}\delta_{tj} = a_{jr}$. This implies, then,
$$T_{ij}' = a_{is}a_{jr}\frac{\partial A_s}{\partial x_r} = a_{is}a_{jr}T_{sr},$$
i.e. a tensor of rank two.

5. First express $\boldsymbol{e}_i'$ in terms of $\boldsymbol{e}_i$:

$$\boldsymbol{e}_1' = \cos\theta\,\boldsymbol{e}_1 + \sin\theta\,\boldsymbol{e}_2, \qquad \boldsymbol{e}_2' = -\sin\theta\,\boldsymbol{e}_1 + \cos\theta\,\boldsymbol{e}_2, \qquad \boldsymbol{e}_3' = \boldsymbol{e}_3.$$

Use the notation $c = \cos\theta$, $s = \sin\theta$ and introduce the matrix $A = (a_{ij})$ by

$$A = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now $T'_{ij} = a_{ir}a_{js}T_rs$, or in matrix notation $T' = ATA^T$, and multiplying out the matrices we find

$$T' = \begin{pmatrix} c^2T_{11} + scT_{12} + scT_{21} + s^2T_{22} & -scT_{11} - s^2T_{21} + c^2T_{12} + scT_{22} & cT_{13} + sT_{23} \\ -scT_{11} + c^2T_{21} - s^2T_{12} + scT_{22} & s^2T_{11} - scT_{12} - scT_{21} + c^2T_{22} & -sT_{13} + cT_{23} \\ cT_{31} + sT_{32} & -sT_{31} + cT_{32} & T_{33} \end{pmatrix}$$

It follows that $T'_{ii} = T'_{11} + T'_{22} + T'_{33} = (c^2 + s^2)(T_{11} + T_{22}) + T_{33} = T_{ii}$, as required.

6. For D_{ij} to be a tensor, then $D'_{ij} = a_{ir}a_{js}D_{rs}$ for *any* transformation to S'. Check if this is the case for the transformation of problem 5. This gives

$$D' = \begin{pmatrix} c^2 + s^2 & 0 & 0 \\ 0 & s^2 + c^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and this is exactly D for all θ. Hence D is not a tensor.

7. First find the transformation of the basis vectors $\boldsymbol{e}_i'$ in terms of $\boldsymbol{e}_i$ to get a matrix A, and then the transformation of $\boldsymbol{e}_i''$ in terms of $\boldsymbol{e}_i'$ to find the matrix B (see problem 5). The matrices are

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now $T' = ATA^T$ and $T'' = BT'B^T = CTC^T$ where $C = BA$, i.e

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

Substituting for the given T and multiplying the matrices gives

$$T'' = \begin{pmatrix} 2a + b & -2a + b & b\sqrt{2} \\ -2a + b & 2a + b & b\sqrt{2} \\ -b\sqrt{2} & -b\sqrt{2} & -2b \end{pmatrix}$$

Now we have

(i) $T''_{ii} = (2a + b) + (2a + b) - 2b = 4a = T_{ii}$.

(ii) $T''_{ij}T''_{ij}$ =sum of the squares of the elements $= 16(a^2 + b^2) = T_{ij}T_{ij}$.

(iii)

$$T''_{ij}T''_{ji} = T''_{11} + T''_{22} + T''_{33} + 2T''_{12}T''_{21} + 2T''_{13}T''_{31} + 2T''_{23}T''_{32} = 16a^2 = T_{ij}T_{ji}$$