UNIVERSITY OF LONDON [E1.11 2003]

B.ENG. AND M.ENG. EXAMINATIONS 2003

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

INFORMATION SYSTEMS ENGINEERING E1.11

MATHEMATICS

Date Wednesday 4th June 2003 10.00 am - 1.00 pm

Answer SEVEN questions

Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

Corrected Copy

[Before starting, please make sure that the paper is complete. There should be SIX pages, with a total of NINE questions. Ask the invigilator for a replacement if this copy is faulty.]

SECTION A **[E1.11 2003]**

1. (i) Express each of the following complex numbers in the form $x + iy$ (with x and y real) :

$$\text{(a)}\ \frac{1+i}{7-i}\ ; \qquad \text{(b)}\ (1+3i)^3\ ; \qquad \text{(c)}\ (1-i)^{17}\ .$$

(ii) Describe what geometrical figure in the complex plane is represented by each of the following equations:

$$\text{(a)}\ |z+1| = |z-1|\ ; \qquad \text{(b)}\ \mathrm{Re}(z^3) = \mathrm{Re}(z)\ .$$

(iii) Find all complex solutions of each of the following equations:

$$\text{(a)}\ \sinh z = 0\ ; \qquad \text{(b)}\ \sinh z + \cosh z = 0\ .$$

The three parts carry, respectively, 40% , 35% and 25% of the marks.

2. (i) Evaluate the following limits:

(a)
$$\lim_{x\to 2} \frac{(x+2)^{1/2} - 2}{x-2}\ ;$$

(b)
$$\lim_{x\to 0}\ x \sin(\tan x)\ ;$$

(c)
$$\lim_{x\to\infty}\ x^{-9}\left\{(x+3)^{10} - (x+1)^{10}\right\}\ .$$

(ii) Differentiate:

(a)
$$\ln\left\{x + (1+x^2)^{1/2}\right\}\ ;$$

(b)
$$(\sin x)^x\ .$$

The two parts carry, respectively, 55% and 45% of the marks.

PLEASE TURN OVER

3. (i) Decide whether each of the following series is convergent or divergent:

$$\text{(a)} \ \sum_{1}^{\infty} \frac{2^n}{n^7} \ ; \qquad \text{(b)} \ \sum_{1}^{\infty} \frac{n+1}{10n+1} \ ; \qquad \text{(c)} \ \sum_{1}^{\infty} \frac{(-1)^n e^n}{n!} \ .$$

(ii) Find the radius of convergence of each of the following power series:

$$\text{(a)} \ \sum_{0}^{\infty} n^3 x^n \ ; \qquad \text{(b)} \ \sum_{0}^{\infty} \frac{n!(n+1)!}{(2n+1)!} \, x^n \ .$$

(iii) Using the Maclaurin series of $\ln(1+x)$ and $\ln(1-x)$ (or otherwise), find the Maclaurin series of the function

$$\ln\left(\sqrt{\frac{1+x}{1-x}}\right) \ .$$

The three parts carry, respectively, 45%, 35% and 20% of the marks.

4. Evaluate the following integrals:

(i)
$$\int \frac{dx}{\sin x} \ ;$$

(ii)
$$\int \frac{x\,dx}{(1-x^2)^{3/2}} \ ;$$

(iii)
$$\int \frac{x^2\,dx}{(1-x^2)^{3/2}} \ ;$$

(iv)
$$\int \frac{2x\,dx}{(x+1)(x^2+1)} \ .$$

The four parts carry, respectively, 15%, 20%, 25% and 40% of the marks.

PLEASE TURN OVER

5. Find the general solution of each of the following differential equations:

(i)
$$\frac{dy}{dx} = (1+x^2)(1+y^2) \; ;$$

(ii)
$$\frac{dy}{dx} + \frac{y}{x} = \sin x \; ;$$

(iii)
$$y'' + 2y' - 3y = e^x \; .$$

(iv) Find the solution of the equation in part (iii) that satisfies the initial conditions $y(0) = y'(0) = 0$.

The four parts carry, respectively, 25%, 25%, 35% and 15% of the marks.

PLEASE TURN OVER

SECTION B

6. Let $A = \begin{pmatrix} -10 & 9 \\ -18 & 17 \end{pmatrix}$.

   (i) Find the eigenvalues and eigenvectors of A.

   (ii) Find an invertible 2×2 matrix P such that $P^{-1}AP$ is a diagonal matrix.

   (iii) Find a 2×2 matrix B such that $B^3 = A$.

   The three parts carry, respectively, 35%, 25% and 40% of the marks.

7. Let $f(x, y) = (x + y)(x^2 + y^2 - 2)$.

   (i) Find the stationary points of $f(x, y)$ and determine their nature.

   (ii) Sketch the contour $f(x, y) = 0$.

   (iii) Sketch some further contours of $f(x, y)$.

   The three parts carry, respectively, 75%, 10% and 15% of the marks.

PLEASE TURN OVER

8. Define $f(x)$ in the interval $0 < x < \pi$ by

$$f(x) \;=\; \begin{cases} \pi & \text{if} \quad 0 < x < \dfrac{\pi}{2}\,, \\[2ex] 0 & \text{if} \quad \dfrac{\pi}{2} \le x < \pi\,. \end{cases}$$

Find

(a) a Fourier cosine series for $f(x)$;

(b) a Fourier sine series for $f(x)$.

Sketch the graph of $f(x)$ in the range $-\pi < x < \pi$ in each case.

Deduce that

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \;=\; \frac{\pi}{4}\,.$$

9. The Heaviside step function $H_a(t)$ is defined by

$$H_a(t) \;=\; \begin{cases} 1 & \text{if} \quad t \ge a\,, \\ 0 & \text{if} \quad t < a\,. \end{cases}$$

Sketch the graph of the function $H_0(t) - H_1(t)$ and find its Laplace transform.

Use the method of Laplace transforms to solve the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y \;=\; H_0(t) - H_1(t) \qquad\qquad (t \ge 0)\,,$$

given $y(0) = y'(0) = 0$.

[*You may use the shift rule:* $L\Big(H_a(t)f(t-a)\Big) = e^{-as}L(f)$] .

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\cos b + \cos a\sin b\,;$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^ng + \binom{n}{1}Df\,D^{n-1}g + \ldots + \binom{n}{r}D^rf\,D^{n-r}g + \ldots + D^nf\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^nf^{(n)}(a)/n! + \epsilon_n(h),$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$