UNIVERSITY OF LONDON [E1.11 2006]

B.ENG. AND M.ENG. EXAMINATIONS 2006

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

INFORMATION SYSTEMS ENGINEERING E1.11

MATHEMATICS

Date Tuesday 30th May 2006 10.00 am - 1.00 pm

Answer ANY SEVEN questions

Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

[Before starting, please make sure that the paper is complete. There should be SIX pages, with a total of NINE questions. Ask the invigilator for a replacement if this copy is faulty.]

SECTION A **[E1.11 2006]**

1. (i) Find all possible values of the following complex numbers.

   Give your answer in the form $x + iy$ (with x and y real) :

   (a) $(1 \; + \; 2i)^2$;

   (b) $\ln(1 \; + \; i)$;

   (c) $(-1 \; + \; i)^{1/3}$;

   (d) $\mathrm{sech}(1 \; + \; i\,\pi/4)$.

   (ii) Find all the solutions of the equation $\tanh z \; = \; 1/2$.

   Give your answer in the form $x + iy$ (with x and y real).

2. (i) Differentiate $y \; = \; (\tan^{-1} x)^{-1}$.

   (ii) Find the stationary points of $y \; = \; x^2\, e^{-x^2}$ and classify them as maxima or minima.

   Sketch the curve.

PLEASE TURN OVER

3. (i) Using l'Hôpital's Rule or otherwise, evaluate the following limits :

(a) $$\lim_{x\to 2} \frac{x^2 - x - 2}{2x^2 + 2x - 12} \ ;$$

(b) $$\lim_{x\to 1} \frac{\ln x}{\cos\left(\frac{\pi x}{2}\right)} \ .$$

(ii) Use standard tests to determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^{n/2}}{\sqrt{n!}}$ converges or diverges.

(iii) Find the interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x^2+1)^n}{3^n\, n^3}$ and investigate the endpoints.

4. Evaluate the following integrals :

(i) $$\int \tan x \, dx \ ;$$

(ii) $$\int x \sec^2 x \, dx \ ;$$

(iii) $$\int \cos^2 (2x) \, dx \ ;$$

(iv) $$\int \frac{dx}{x(x^2 + 2x + 1)} \ .$$

5. (i) Find the general solution of the 1st order differential equation

$$\frac{dy}{dx} + (\ln x)\, y = x\, e^{-x \ln x} .$$

(ii) Find the solution of the 2nd order differential equation

$$\frac{d^2y}{dx^2} - 8\,\frac{dy}{dx} + 16\, y = e^{4x} ,$$

with $y = 1,\ \dfrac{dy}{dx} = 0$ at $x = 0$.

SECTION B

6. (i) Let

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 3 & -1 & 4 \\ 1 & -1 & 2 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Use Gaussian elimination to find all solutions of the system of simultaneous equations

$$Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(ii) Find a condition on the entries of a column vector $c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

which ensures that the system $Ax = c$ has no solutions, where A is the matrix in (i).

(iii) Let $B = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.

Find an invertible 2×2 matrix P such that $P^{-1}BP$ is a diagonal matrix.

7. (i) Let $u = x^2 + y^2$, $v = xy$ and $f = f(u, v)$.

Express $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ in terms of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

(ii) Let $g(x, y) = x^3y + xy^2 - xy$.

Find the six stationary points of g and determine whether they are maxima, minima or saddle points.

8. Sketch the graph of the function

$$f(x) \;=\; |\,\sin x\,|\;.$$

Calculate the Fourier series for $f(x)$, giving the general term.

Deduce that

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \;\ldots\; = \frac{\pi}{4} - \frac{1}{2}$$

and

$$\frac{1}{3\cdot 5} - \frac{1}{7\cdot 9} + \frac{1}{11\cdot 13} - \frac{1}{15\cdot 17} + \;\ldots\; = \frac{\pi}{4\sqrt{2}} - \frac{1}{2}\,.$$

9. The Laplace transform of a function $f(t)$ is defined by

$$\mathcal{L}(f(t)) \;=\; F(s) \;=\; \int_0^\infty e^{-st}\, f(t)\, dt\;.$$

(i) Find $\mathcal{L}(e^{-t}\,\sin t)$.

(*Any rule used must be proved.*)

(ii) Use Laplace transforms to find functions x, y of t satisfying the following simultaneous differential equations :

$$\frac{dx}{dt} + \frac{dy}{dt} + x = 0\,,$$

$$\frac{dx}{dt} + 2\,\frac{dy}{dt} - x = e^{-t}\,,$$

with $x(0) = \frac{1}{2},\;\; y(0) = 0\,.$

[*You may assume that* $\mathcal{L}(f'(t)) \;=\; -\,f(0) \;+\; s\,\mathcal{L}(f(t))$.]

END OF PAPER

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course ISE 1.6
Question 1		Marks & seen/unseen

Parts

(a)(i) $(1+2i)^2 = 1+4i+4i^2 = -3+4i$ — 1

(ii) $\ln(1+i) = \ln\left(\sqrt{2}e^{i\frac{\pi}{4}+i2n\pi}\right) = \ln\sqrt{2} + \frac{i\pi}{4} + i2n\pi$ — 3

$$= \frac{1}{2}\ln 2 + i\left(\frac{\pi}{4}+2n\pi\right)$$

(iii) $(-1+i)^{1/3} = \left(\sqrt{2}e^{i\frac{3\pi}{4}+i2n\pi}\right)^{1/3}$

$$= 2^{1/6}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2^{-1/3}(1+i)$$ — 4

$$= 2^{1/6}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

$$= 2^{1/6}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$$

(iv) $\operatorname{sech}\left(1+\frac{i\pi}{4}\right) = \dfrac{1}{\cosh\left(1+\frac{i\pi}{4}\right)} = \dfrac{2}{e^{1+\frac{i\pi}{4}} + e^{-1-\frac{i\pi}{4}}}$

$$= \frac{2}{e\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)+\frac{1}{e}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)} = \frac{2\sqrt{2}e}{e^2(1+i)+(1-i)} = \frac{2\sqrt{2}e}{(e^2+1)+i(e^2-1)}$$ — 6

$$= \frac{2\sqrt{2}e\left[(e^2+1)-i(e^2-1)\right]}{(e^2+1)^2+(e^2-1)^2} = \frac{2\sqrt{2}e\left[(e^2+1)-i(e^2-1)\right]}{2e^4+2} = \frac{\sqrt{2}e(e^2+1)}{e^4+1} - i\frac{\sqrt{2}e(e^2-1)}{e^4+1}$$

(b) $\tanh z = \frac{1}{2} \Rightarrow \frac{\sinh z}{\cosh z} = \frac{1}{2} \Rightarrow \frac{e^z - e^{-z}}{2} = \frac{e^z + e^{-z}}{4}$

$\Rightarrow 3e^{-z} = e^z \quad \Rightarrow e^{2z} = 3e^{i2n\pi}$ — 6

$$\Rightarrow 2z = \ln 3 + i2n\pi$$

$$\Rightarrow z = \frac{1}{2}\ln 3 + in\pi$$

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	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course ISE 1.6
Question 2		Marks & seen/unseen

Parts

(a) $y = (\tan^{-1} x)^{-1}$

$\Rightarrow \dfrac{dy}{dx} = -(\tan^{-1} x)^{-2} \dfrac{d}{dx}(\tan^{-1} x)$

Let $v = \tan^{-1} x \Rightarrow \tan v = x$

$\Rightarrow \sec^2 v \dfrac{dv}{dx} = 1$

$\Rightarrow \dfrac{dv}{dx} = \dfrac{1}{\sec^2 v} = \dfrac{1}{1+\tan^2 v} = \dfrac{1}{1+x^2}$ 8

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-1}{(\tan^{-1} x)^2} \cdot \dfrac{1}{1+x^2}$

(b) $y = x^2 e^{-x^2}$

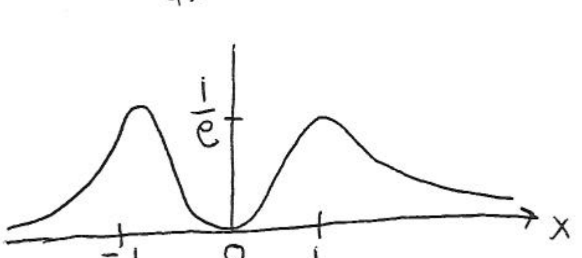
$\Rightarrow \dfrac{dy}{dx} = 2x e^{-x^2} - 2x^3 e^{-x^2} = 2x(1-x^2)e^{-x^2}$

Stationary points where $\dfrac{dy}{dx} = 0 \Rightarrow x = 0, \pm 1$ 9

$\dfrac{d^2y}{dx^2} = 2e^{-x^2} - 4x^2 e^{-x^2} - 6x^2 e^{-x^2} + 4x^4 e^{-x^2} = (2 - 10x^2 + 4x^4)e^{-x^2}$

At $x = \pm 1$, $\dfrac{d^2y}{dx^2} = (2-10+4)e^{-1} = -4e^{-1} < 0 \Rightarrow$ maximum

At $x = 0$, $\dfrac{d^2y}{dx^2} = 2 > 0 \Rightarrow$ minimum



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	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course ISE 1.6
Question 3		Marks & seen/unseen

Parts

(a) (i) $\lim_{x \to 2} \frac{x^2 - x - 2}{2x^2 + 2x - 12} = \lim_{x \to 2} \frac{2x-1}{4x+2} = \frac{3}{10}$ — 2

(ii) $\lim_{x \to 1} \frac{\ln x}{\cos\left(\frac{\pi x}{2}\right)} = \lim_{x \to 1} \frac{1/x}{-\frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right)} = -\frac{2}{\pi}$ — 3

(b) $\sum_{n=1}^{\infty} \frac{e^{n/2}}{\sqrt{n!}} \qquad a_n = \frac{e^{n/2}}{\sqrt{n!}}$

$\rho = \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \left(\frac{e^{\frac{n+1}{2}}}{\sqrt{(n+1)!}} \frac{\sqrt{n!}}{e^{n/2}}\right)$ — 6

$= \lim_{n \to \infty} \left(\frac{e^{1/2}}{\sqrt{n+1}}\right) = 0 \Rightarrow$ convergent.

(c) $\sum_{n=1}^{\infty} \frac{(x^2+1)^n}{3^n n^3} \qquad a_n = \frac{(x^2+1)^n}{3^n n^3}$

$\rho = \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \left(\frac{(x^2+1)^{n+1}}{3^{n+1}(n+1)^3} \frac{3^n n^3}{(x^2+1)^n}\right)$

$= \lim_{n \to \infty} \left(\frac{x^2+1}{3} \frac{1}{(1+\frac{1}{n})^3}\right) = \frac{x^2+1}{3}$ — 9

Convergent for $\frac{x^2+1}{3} < 1 \Rightarrow x^2 < 2 \Rightarrow -\sqrt{2} < x < \sqrt{2}$

Endpoints: at $x = \pm\sqrt{2}$, series becomes:

$\sum_{n=1}^{\infty} \frac{((\sqrt{2})^2+1)^n}{3^n n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ which is convergent.

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EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course: ISE1.6

Question: 4

Marks & seen/unseen

Parts

(a) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln(\cos x) + c$ — 3

(b) $\int x \sec^2 x \, dx$ $\quad u = x \quad \frac{dv}{dx} = \sec^2 x$

$\frac{du}{dx} = 1 \quad v = \tan x$ — 3

$= x \tan x - \int \tan x \, dx$

$= x \tan x + \ln(\cos x) + c$

(c) $\int \cos^2(2x) \, dx$

$= \int \frac{1}{2}(1 + \cos 4x) \, dx$ — 4

$= \frac{1}{2}x + \frac{1}{8}\sin 4x + c$

(d) $\int \frac{dx}{x(x^2+2x+1)}$

$\frac{1}{x(x^2+2x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+1} \Rightarrow 1 = A(x^2+2x+1) + (Bx+C)x$

$\Rightarrow 1 = x^2(A+B) + x(2A+C) + A$ — 10

$\Rightarrow A = 1, B = -1, C = -2$

$= \int \left\{ \frac{1}{x} - \frac{x+2}{x^2+2x+1} \right\} dx = \int \left\{ \frac{1}{x} - \frac{x+1+1}{(x+1)^2} \right\} dx$

$= \int \left\{ \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} \right\} dx = \ln x - \ln(x+1) + \frac{1}{x+1} + c$

$= \ln \frac{x}{x+1} + \frac{1}{x+1} + c$

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	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course ISE 1.6
Question 5		Marks & seen/unseen

Parts

(a) $\frac{dy}{dx} + (\ln x)y = xe^{-x\ln x}$

Integrating factor $e^{\int \ln x\, dx}$

$\int \ln x\, dx$ $\quad u = \ln x \quad \frac{dv}{dx} = 1 \quad \int \ln x\, dx = x\ln x - \int dx = x\ln x - x$

$\frac{du}{dx} = \frac{1}{x} \quad v = x$

$\Rightarrow \frac{d}{dx}\left(ye^{x\ln x - x}\right) = xe^{-x\ln x}e^{x\ln x - x} = xe^{-x}$ 10

$\Rightarrow ye^{x\ln x - x} = \int xe^{-x}$ $\quad u = x \quad \frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = 1 \quad v = -e^{-x}$

$= -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x} + c$

$\Rightarrow y = e^{-x\ln x}\left(-1 - x + ce^{x}\right)$

(b) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = e^{4x}$

Auxiliary equation: $\lambda^2 - 8\lambda + 16 = 0 \Rightarrow (\lambda - 4)^2 = 0$

C.F. $y = (Ax+B)e^{4x}$

P.I. $y = Cx^2e^{4x}$ 10

$\Rightarrow 16Cx^2e^{4x} - 8\left(4Cx^2e^{4x} + 2Cxe^{4x}\right) + \left(2Ce^{4x} + 16Cxe^{4x} + 16Cx^2e^{4x}\right) = e^{4x}$

$\Rightarrow C = \frac{1}{2}$ $\quad$ Full solution: $y = (Ax+B)e^{4x} + \frac{1}{2}x^2e^{4x}$

$y = 1$ at $x = 0 \Rightarrow B = 1$

$\frac{dy}{dx} = 0$ at $x = 0 \Rightarrow 0 = xe^{4x} + 2x^2e^{4x} + Ae^{4x} + 4(Ax+1)e^{4x}$ at $x = 0$

$\Rightarrow A = -4$.

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$\Rightarrow y = (1-4x)e^{4x} + \frac{1}{2}x^2e^{4x}$.

EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course: ISE 1.

Question: 6, solution.

Marks & seen/unseen

Parts

(i) Augmented matrix

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 3 & -1 & 4 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right) \rightarrow \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

SIMILAR EGS SEEN

New system: $x_1 - x_2 + 2x_3 = 0$
$x_2 - x_3 = 0$

General solution is $x = (-a, a, a)$ (any a). 6

(ii)
$$\begin{pmatrix} 2 & -1 & 3 & c_1 \\ 3 & -1 & 4 & c_2 \\ 1 & -1 & 2 & c_3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 2 & c_3 \\ 0 & 1 & -1 & c_1 - 2c_3 \\ 0 & 2 & -2 & c_2 - 3c_3 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & -1 & 2 & c_3 \\ 0 & 1 & -1 & c_1 - 2c_3 \\ 0 & 0 & 0 & c_2 - 3c_3 - 2(c_1 - 2c_3) \end{pmatrix}$$

System has no solutions when constant on RHS of last equation is nonzero, i.e. when

$$c_2 - 3c_3 - 2(c_1 - 2c_3) \neq 0$$

i.e. $\underline{2c_1 - c_2 - c_3 \neq 0}$ 6

(iii) Characteristic poly of B is $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix}$ SIMILAR SEEN

$= \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1).$

So evalues are $4, -1$. 2

$\underline{\lambda = 4}$ Evectors $\left(\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right) \rightarrow a\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ 2

$\underline{\lambda = -1}$ Evecs $\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right) \rightarrow b\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$ 2

Setter's initials: MLL

Checker's initials: MJA

Parts

So $\underline{P = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}}$ will do 2

Parts

(i)

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot 2x + \frac{\partial f}{\partial v} \cdot y \quad ①$$

UNSEEN

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot 2y + \frac{\partial f}{\partial v} \cdot x \quad ②$$

2

① × y − ② × x gives

$$y f_x - x f_y = f_v \left(y^2 - x^2\right)$$

$$\therefore \quad f_v = \frac{y f_x - x f_y}{y^2 - x^2}$$

2

① × x − ② × y gives

$$x f_x - y f_y = f_u \left(2x^2 - 2y^2\right)$$

$$\therefore \quad f_u = \frac{x f_x - y f_y}{2(x^2 - y^2)}$$

2

(ii) Here

SIMILAR SEEN

$$① \quad g_x = 3x^2 y + y^2 - y = y\left(3x^2 - 1 + y\right)$$

$$② \quad g_y = x^3 + 2xy - x = x\left(x^2 - 1 + 2y\right)$$

2

For stationary point, $g_x = g_y = 0$.

From ①, $y = 0$ or $3x^2 - 1 + y = 0$

From ②, $x = 0$ or $x^2 - 1 + 2y = 0$.

Question. 7, ctd. | Marks & seen/unseen

Parts

If $y = 0$ then $x = 0$ or ± 1.

If $x = 0$ then $y = 0$ or 1.

If $3x^2 - 1 + y = x^2 - 1 + 2y = 0$, then

sub. $y = 1 - 3x^2$, giving

$$x^2 - 1 + 2(1 - 3x^2) = 0$$

$$\Rightarrow 5x^2 = 1$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{5}}, \quad y = 1 - \frac{3}{5} = \frac{2}{5}.$$

Hence get 6 stationary points

$$(0,0),\ (1,0),\ (-1,0),\ (0,1),\ \left(\tfrac{1}{\sqrt{5}}, \tfrac{2}{5}\right), \left(-\tfrac{1}{\sqrt{5}}, \tfrac{2}{5}\right)$$ 6

Nature Now

$$g_{xx} = 6xy, \quad g_{xy} = 3x^2 + 2y - 1, \quad g_{yy} = 2x.$$ 2

Hence, for $\Delta = g_{xy}^2 - g_{xx}\, g_{yy}$, get

pt.	$(0,0)$	$(1,0)$	$(-1,0)$	$(0,1)$	$(\frac{1}{\sqrt{5}}, \frac{2}{5})$	$(-\frac{1}{\sqrt{5}}, \frac{2}{5})$
Δ	1	4	4	1	$-\frac{4}{5}$	$-\frac{4}{5}$
g_{xx}					>0	<0

2

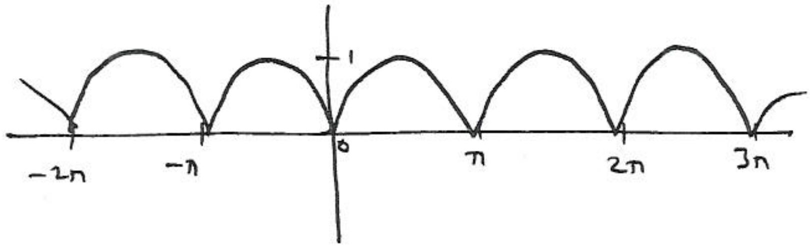
Hence $(0,0)$, $(\pm 1, 0)$, $(0,1)$ are saddles

$(\frac{1}{\sqrt{5}}, \frac{2}{5})$ is a minimum

$(-\frac{1}{\sqrt{5}}, \frac{2}{5})$ is a maximum 2

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course ISE 1.
Question. 8, solution		Marks & seen/unseen
Parts		SIMILAR SEEN

Graph:



3

Fourier series: is a cosine series (even function)

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x \, dx = \frac{2}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{4}{\pi}$$

and for $n \geq 1$,

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \left(\sin(n+1)x - \sin(n-1)x\right) dx$$

For $n = 1$, this is zero. For $n \neq 1$

$$= \frac{1}{\pi}\left[\frac{1}{n-1}\cos(n-1)x - \frac{1}{n+1}\cos(n+1)x\right]_0^{\pi}$$

$$= \begin{cases} 0, & n \text{ odd} \\ \dfrac{-4}{\pi(n^2-1)}, & n \text{ even.} \end{cases}$$

So Fourier series is

$$\frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{2^2-1}\cos 2x + \frac{1}{4^2-1}\cos 4x + \ldots\right)$$

9

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos 2nx.$$

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EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course: ISE 1.

Question. 8 ctd.

Marks & seen/unseen

Parts

Put $x = \frac{\pi}{2}$:

$$\sin\frac{\pi}{2} = 1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$$

$$\therefore \quad \frac{4}{\pi}\left(\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots\right) = 1 - \frac{2}{\pi}$$

$$\therefore \quad \underline{\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots = \frac{\pi}{4} - \frac{1}{2}}$$

4

Put $x = \frac{\pi}{4}$:

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{-1}{3\cdot 5} + \frac{1}{7\cdot 9} - \frac{1}{11\cdot 13} + \dots\right)$$

$$\therefore \quad \frac{1}{3\cdot 5} - \frac{1}{7\cdot 9} + \frac{1}{11\cdot 13} - \dots$$

$$= \frac{\pi}{4}\left(\frac{1}{\sqrt{2}} - \frac{2}{\pi}\right)$$

$$= \underline{\frac{\pi}{4\sqrt{2}} - \frac{1}{2}}.$$

4

Setter's initials: MLR

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	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course ISE 1.
Question. 9, solution.		Marks & seen/unseen

Parts

(i) Let $f(t) = e^{-t}\sin t$. Then

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-(s+1)t}\sin t\,dt$$

$$= \left[-\frac{1}{s+1}e^{-(s+1)t}\sin t\right]_0^\infty + \int_0^\infty \frac{1}{s+1}e^{-(s+1)t}\cos t\,dt$$

$$= 0 + \frac{1}{s+1}\left[-\frac{1}{s+1}e^{-(s+1)t}\cos t\right]_0^\infty - \frac{1}{s+1}\int_0^\infty \frac{1}{s+1}e^{-(s+1)t}\sin t\,dt$$

$$= \frac{1}{(s+1)^2} - \frac{1}{(s+1)^2}F(s)$$

$$\therefore F(s)\left(1 + \frac{1}{(s+1)^2}\right) = \frac{1}{(s+1)^2}$$

$$\therefore F(s) = \frac{1}{(s+1)^2+1}$$

SIMILAR SEEN

6

(ii) Take Laplace transforms:

① $-\frac{1}{2} + s\mathcal{L}(x) + s\mathcal{L}(y) + \mathcal{L}(x) = 0$

② $-\frac{1}{2} + s\mathcal{L}(x) + 2s\mathcal{L}(y) - \mathcal{L}(x) = \frac{1}{s+1}$

4

Setter's initials MLR	Checker's initials	Page number 12

Parts

So

① $(s+1)\mathcal{L}(x) + s\mathcal{L}(y) = \frac{1}{2}$

② $(s-1)\mathcal{L}(x) + 2s\mathcal{L}(y) = \frac{1}{s+1} + \frac{1}{2} = \frac{s+3}{2(s+1)}$.

Then $2\times$① $-$ ② gives

$$(s+3)\mathcal{L}(x) = 1 - \frac{s+3}{2s+2} = \frac{s-1}{2s+2}$$

$$\therefore \mathcal{L}(x) = \frac{s-1}{2(s+1)(s+3)}$$

$$= \frac{1}{2}\left(\frac{2}{s+3} - \frac{1}{s+1}\right)$$

$$\therefore \underline{x = e^{-3t} - \tfrac{1}{2}e^{-t}}$$ 5

From ①, $s\mathcal{L}(y) = \frac{1}{2} - (s+1)\mathcal{L}(x)$

$$= \frac{1}{2} - \frac{s-1}{2(s+3)} = \frac{2}{s+3}$$

So $\mathcal{L}(y) = \frac{2}{s(s+3)} = \frac{2}{3}\left(\frac{1}{s} - \frac{1}{s+3}\right)$

$$\therefore \underline{y = \frac{2}{3}\left(1 - e^{-3t}\right)}$$ 5