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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

MSc and EEE PART IV: MEng and ACGI

ESTIMATION AND FAULT DETECTION

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : R.B. Vinter
Second Marker(s) : J.C. Allwright

Information for candidates:

Some formulae relevant to the questions.

The normal $N(m, \sigma^2)$ density:

$$p(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-m)^2}{2\sigma^2}\right)$$

System equations:

$$\begin{aligned} x_k &= F x_{k-1} + u^s + w_k \\ y_k &= H x_k + u^o + v_k \,. \end{aligned}$$

Here, w_k and v_k are white noise sequences with covariances Q^s and Q^0 respectively.

The Kalman filter equations are

$$\begin{aligned} P_{k|k-1} &= F P_{k-1|k-1} F^T + Q^s \\ P_k &= P_{k|k-1} - P_{k|k-1} H^T (H P_{k|k-1} H^T + Q^o)^{-1} H P_{k|k-1} \,, \\ K_k &= P_{k|k-1} H^T (H P_{k|k-1} H^T + Q^o)^{-1} \,, \\ \hat{x}_k &= \hat{x}_{k|k-1} + K_k (y_k - \hat{y}_{k|k-1}) \,, \end{aligned}$$

in which $\hat{x}_{k|k-1} = F\hat{x}_{k-1} + u^s$ and $\hat{y}_{k|k-1} = H\hat{x}_{k|k-1} + u^o$

1. Consider the stochastic differential equation

$$\ddot{y}(t) \;=\; w(t)$$

where $\{w_t\}$ is Gaussian white noise with $E[w(t)w(s)] \;=\; \delta(t-s)$.

(i): Show that

$$\dot{y}(t) \;=\; \dot{y}(0) + \textstyle\int_0^t w(s)ds \quad \text{and} \quad y(t) \;=\; y(0) + \dot{y}(0)t + \int_0^t \int_0^s w(s')ds'ds\ .$$

Hence show that $x(t) = (x_1(t), x_2(t))^T = (y(t), \dot{y}(t))^T$ satisfies

$$x(t) \;=\; \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0) + \int_0^t \begin{bmatrix} t-s \\ 1 \end{bmatrix} w(s)ds\ .$$

Hint: Use the integration by parts formula to evaluate the double integral. [8]

(ii): Now assume that $x(0)$ is independent of $\{w_t\}$. Derive a formula for

$$P_t = \text{cov}\,\{x(t)\}$$

in terms of P_0 and t. [8]

(iii): Finally, assume that $x(0) = 0$. Show that the correlation coefficient of $x_1(t)$ and $x_2(t)$, namely

$$\rho(x_1(t), x_2(t)) \;=\; \frac{E[\,x_1(t)\,x_2(t)\,]}{(Ex_1^2(t))^{\frac{1}{2}}\,(Ex_2^2(t))^{\frac{1}{2}}}\ ,$$

is a constant. [4]

2. A sensor is believed to be at the origin in one-dimensional space. The sensor has a random time-varying bias b_k governed by the auto-regressive model

$$b_k - ab_{k-1} = w_k \ .$$

Here $\{w_k\}$ is a white noise sequence for which $w_k \sim N(0, \sigma_b^2)$. a and σ_b^2 are a known constants, $-1 < a < 1$.

The observation y_k at time k is of an unknown fixed point r_0 on the real line, corrupted by white noise:

$$y_k = r_0 - b_k + v_k$$

where $v_k \sim N(0, \sigma^2)$.

(i): Formulate the problem of simultaneously estimating the position and the bias (r_0, b_k), at time k, as a standard Kalman filtering problem:

$$\begin{aligned} x_k &= Fx_{k-1} + \tilde{w}_k \\ y_k &= h^T x_k + v_k \ . \end{aligned}$$

What are F, h^T and $cov\{\tilde{w}_k\}$? [6]

(ii): Is (F, h^T) observable? [2]

(iii): By solving the algebraic Riccati equation determine the steady state predictor error covariance

$$S = \{s_{ij}\} = lim_{k\to\infty} P_{k|k-1}$$

where

$$P_{k|k-1} = cov\{x_k \,|\, y_{1:k-1}\} \ .$$

Comment on the values of s_{11}. Would it be sensible to use the steady state version of the filter, in place of the 'optimal' time-varying linear least squares filter? [12]

3a: Take two jointly distributed, scalar, random variables x and v with mean m_x and m_v respectively. Denote by $\rho(x,v)$ the correlation coefficient:

$$\rho(x,v) \;=\; \frac{E[(x-m_x)(v-m_v)]}{(E(x-m_x)^2)^{\frac{1}{2}}\;(E(v-m_v)^2)^{\frac{1}{2}}}\;.$$

Show that, for $j = 1, 2$,

$$\rho = (-1)^j \quad \text{implies} \quad \sigma_x^{-1}(x-m_x) \;=\; (-1)^j \times \sigma_v^{-1}(v-m_v)\;\;.$$

Hint: Calculate $E[\,|\sigma_x^{-1}(x-m_x) - (-1)^j\sigma_v^{-1}(v-m_v)|^2]$. [5]

3b. A noisy scalar measurement y is taken of a signal x. x is modelled as a scalar random variable. y is taken to be x corrupted by additive correlated noise:

$$y = x + v\;.$$

Here $E[x] = m_x$, $\text{cov}\{x\} = \sigma_x^2$, $E[v] = 0$, $\text{cov}\{v\} = \sigma_v^2$ and $\text{cov}\{x,v\} = \rho\sigma_x\sigma_v$, for some constants m_x, $\sigma_x^2 > 0$, $\sigma_v^2 > 0$ and ρ, $-1 \le \rho \le +1$.

(i): Calculate the linear least squares estimate $\hat{x}$ of x given y, and the mean square estimation error $E[|x-\hat{x}|^2]$. [5]

(ii): Suppose that $\sigma_x \neq \sigma_v$. What values of ρ minimize the mean square error? [5]

(iii): Suppose that $\sigma_x = \sigma_v$ and $\rho(x,v) = -1$. What is the mean square estimation error in this case? Comment on your answer. [5]

Hint: In part b(iii), use your answer to part a.

4. N identical sensors are used to take independent measurements of the position x of an object in one dimensional space. The k'th sensor measurement y_k is related to x according to:

$$y_k = x + e_k .$$

Assume that the additive noise terms $e_1, \ldots, e_N$ and x are independent random variables and

$$E[x] = 0,\ E[e_1] = \ldots = E[e_N] = 0 ,$$
$$\text{var}\{x\} = \sigma_x^2,\ \text{var}\{e_1\} = \ldots = \text{var}\{e_N\} = \sigma_e^2 .$$

(i): Derive the linear least squares estimate of $\hat{x}$ given $y_1, \ldots, y_N$. [8]

(ii): Derive the mean square estimation error $E|x - \hat{x}|^2$. [7]

(iii): Suppose that $\sigma_x^2 = 0.5$ cm^2 and $\sigma_e^2 = 1$ cm^2. It is required that the mean square estimation error satisfies:

$$E|x - \hat{x}|^2 \leq 0.01\,\text{cm}^2 .$$

What is the minimum number of sensors for which this constraint is satisfied? [5]

Hint: Derive the linear least squares estimate by direct minimization of the mean square error, and not by using the standard formula for the linear least squares estimator. You can use the fact that, by symmetry, the weights in the linear least squares estimator are all the same.

5. N independent measurements y_k are taken of the composition of liquid in a tank, to decide whether biological contamination has occurred. Two hypotheses are considered:

(H_0) : contamination has not occurred. In this case, $y_k \sim N(0, \sigma^2)$

(H_1) : contamination has occurred. In this case, $y_k \sim N(a^k, \sigma^2)$.

Here, σ^2 and a are known positive constants. The situation when a test selects (H_1) when (H_0) is true is called a *false alarm*.

Let $l(y_1, \ldots, y_N)$ be the log likelihood ratio:

$$l(y_1, \ldots, y_N) = \log_e \frac{p_1(y_1, \ldots, y_N)}{p_0(y_1, \ldots, y_N)} .$$

In this formula, p_j is the joint density of $(y_1, \ldots, y_N)$ under hypothesis (H_j), $j = 0, 1$.

(i): Show that the log likelihood ratio is [8]

$$l(y_1, \ldots, y_N) = \sigma^{-2} \sum_{k=1}^{N} a^k \left[y_k - \frac{1}{2} a^k \right] .$$

(ii): Assuming (H_0) (no contamination), calculate the probability density of $l(y_1, \ldots, y_N)$. [4]

(iii): Taking (H_0) as the null hypothesis, construct a Neyman Pearson test of whether contamination has occurred, at the 0.01 significance level, i.e. under the constraint that the probability of a false alarm is 0.01. [8]

6a. Signal and observation processes are described by the equations

$$\begin{aligned} x_k &= f(x_{k-1}) + w_k \\ y_k &= h(x_k) + v_k, \end{aligned}$$

in which w_k and v_k are white noise sequences with covariances Q^s and Q^0. f and h are given (possibly nonlinear) functions.

By making suitable linear approximations to the above nonlinear equations, derive the standard extended Kalman filter equations for estimating the conditional mean and covariance of x_k given $y_{1:k}$, taking as starting point the Kalman filter for linear, Gaussian estimation. [6]

What form does the measurement process matrix H in the extended Kalman filter equations for 'range only tracking', i.e. when the state variable is two-dimensional and

$$h(x_1, x_2) = \left(x_1^2 + x_2^2\right)^{\frac{1}{2}} \ ?$$

[4]

6b. Consider stationary processes $\{x_k\}$ and $\{y_k\}$ associated with the state space model

$$\begin{aligned} x_k &= \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x_{k-1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e_k \\ y_k &= [c_0 \; c_1] \, x_k \, . \end{aligned}$$

In these equations e_k is a scalar, unit variance white noise process. Suppose that the spectral density of y_k is

$$\Phi_y(\omega) = \frac{1}{2} \times \frac{1 - \frac{4}{5}\cos \omega t}{1 + \frac{3}{5}\cos \omega t} \, .$$

Determine consistent values of the parameters a_0 and a_1. [10]

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1(i) $\ddot{y}(t) = w(t)$. Integrating across this equation gives

$$\dot{y}(t) = \dot{y}(0) + \int_0^t \ddot{y}(s)ds = \dot{y}(0) + \int_0^t w(s)ds$$

A further integration gives

$$y(t) = y(0) + \dot{y}(0)t + \int_0^t \int_0^s w(s')ds'ds$$

Parts integration gives

$$\int_0^t 1 \times \int_0^s w(s')ds'ds = t.\int_0^t w(s)ds - \int_0^t s\, w(s)ds.$$ Hence

$$y(t) = y(0) + \dot{y}(0)t + \int_0^t (t-s)w(s)ds.$$

In vector notation:

$$x(t) = \underbrace{\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}}_{F(t)} x(0) + \int_0^t \underbrace{\begin{bmatrix} t-s \\ 1 \end{bmatrix}}_{b(t-s)} w(s)ds \qquad [8]$$

(ii) Since $x(0)$ and $w(s)$, $s < t$ are independent

$$\text{cov}\{x(t)\} = F(t)P_0F^T(t) + \int_0^t b(t-s)b^T(t-s)ds$$

$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} P_0 \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} + A(t),$$

where

$$A(t) = \int_0^t \begin{bmatrix} (t-s)^2 & (t-s) \\ (t-s) & 1 \end{bmatrix} ds = \begin{bmatrix} \frac{1}{3}t^3 & \frac{1}{2}t^2 \\ \frac{1}{2}t^2 & t \end{bmatrix}$$

[8]

(iii) If $P_0 = 0$

$$\text{cov}\{x(t)\} = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t^3 & \frac{1}{2}t^2 \\ \frac{1}{2}t^2 & t \end{bmatrix}$$

The correlation coefficient is

$$\rho(x_1(t), x_2(t)) = \frac{P_{12}}{\sqrt{P_{11}P_{22}}} = \frac{\frac{1}{2}t^2}{\sqrt{\frac{1}{3}t^4}} = \frac{\sqrt{3}}{2},$$

a constant, as claimed. [4]

2(i) Take $(x_k^1, x_k^2) = (r_0, b_k)$. Since x_k^1 does not change, $x_k^1 = x_{k-1}^1$. We know also $x_k^2 = b_k = a b_{k-1} + w_k = a x_{k-1}^2 + w_k$. Also, $y_k = r_0 - b_k + v_k = [1\ -1]\, x_k + v_k$. In matrix form: $x_k = F x_{k-1} + \tilde{w}_k$ and $y_k = h^T x_k + v_k$, and $\tilde{w}_k = \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_k$ where $F = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$, $h^T = [1\ -1]$.

We have $\operatorname{cov}\{\tilde{x}_k\} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0\ 1] \operatorname{cov}\{w_k\} = \overbrace{\begin{bmatrix} 0 & 0 \\ 0 & \sigma_b^2 \end{bmatrix}}^{Q^s}$. [6]

(ii) The observability matrix is $\begin{bmatrix} h^T \\ h^T F \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -a \end{bmatrix}$. This is non-singular (and so (F, h^T) is observable) since $a \neq 1$. [2]

(iii) The ARE is $S = FSF^T - FSh(h^T S h + \sigma^2)^{-1} h^T S F + Q^s$, or

$$\begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} = \begin{bmatrix} s_{11} & a s_{12} \\ a s_{12} & a^2 s_{22} \end{bmatrix} - \frac{1}{(s_{11} - 2s_{12} + s_{22} + \sigma^2)} \begin{bmatrix} (s_{11}-s_{12})^2 & a(s_{11}-s_{12})(s_{12}-s_{22}) \\ a(s_{11}-s_{12})(s_{12}-s_{22}) & a^2(s_{12}-s_{22})^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_b^2 \end{bmatrix}$$

Equating entries of these matrices gives:

$s_{11} = s_{11} - \dfrac{(s_{11}-s_{12})^2}{s_{11} - 2s_{12} + s_{22} + \sigma^2}$. This implies $s_{11} = s_{12}$

$s_{12} = a s_{12} - \dfrac{a(s_{11}-s_{12})(s_{12}-s_{22})}{s_{11} - 2s_{12} + s_{22} + \sigma^2}$. This implies $s_{12} = 0$. Hence $s_{11} = 0$

Then $s_{22} = a^2 s_{22} - \dfrac{a^2 s_{22}^2}{s_{22} + \sigma^2} + \sigma_b^2$

This gives $s_{22} = \sqrt{\left(\dfrac{\sigma^2(1-a^2) - \sigma_b^2}{(1-a^2)^2}\right)^2 + \dfrac{4\sigma^2}{1-a^2}} - \dfrac{\sigma^2(1-a^2) - \sigma_b^2}{(1-a^2)}$.

We see that $s_{22} > 0$, while $s_{11} = s_{12} = 0$.

"$s_{11} = 0$" tells us that, asymptotically, the mean square prediction error is zero. In other words, the filter determines $\hat{x}_k^1$ exactly in the limit. This is a consequence of the fact that the system noise covariance is zero.

The steady state Kalman filter equations give $\hat{x}_k^1 = \hat{x}_k^1 + 0$

This would not be a sensible filter to choose, because it takes no account of the measurements and does not coincide with $\lim_{k \to \infty} \hat{x}_{k|k}$. [12]

3(a) $E\left[\left|\frac{x-m_x}{\sigma_x} - (-1)^j \frac{v-m_v}{\sigma_v}\right|^2\right] = \frac{\text{cov}\{x\}}{\sigma_x^2} - 2(-1)^j \frac{\text{cov}\{x,v\}}{\sigma_x\sigma_v} + \frac{\text{cov}\{v\}}{\sigma_v^2}$

$$= 1 - 2(-1)^j \rho(x,v) + 1 = \begin{cases} 0 & \text{if } j=0 \text{ and } \rho(x,v)=1 \\ 0 & \text{if } j=1 \text{ and } \rho(x,v)=-1. \end{cases}$$

i.e. $\rho = (-1)^j$ implies $\frac{x-m_x}{\sigma_x} = (-1)^j \frac{v-m_v}{\sigma_v}$ for $j=0,1$. [5]

b(i) $m_y = m_x + 0$, $\text{cov}\{x,y\} = E[(x-x_m)(x-x_m+v)] = \sigma_x^2 + \rho\sigma_x\sigma_v$
and

$$\text{cov}\{y\} = E[(x-x_m+v)^2] = \sigma_x^2 + 2\rho\sigma_x\sigma_v + \sigma_v^2$$

From the standard formulae, the linear least squares estimate is

$$\hat{x} = m_x + \frac{\sigma_x^2\left(1+\rho\left(\sigma_v/\sigma_x\right)\right)}{\sigma_x^2\left(1+2\rho\left(\sigma_v/\sigma_x\right)+\left(\sigma_v/\sigma_x\right)^2\right)}(y-m_x)$$

and

$$E[|x-\hat{x}|^2] = \sigma_x^2\left[1 - \frac{(1+\alpha\rho)^2}{(1+2\alpha\rho+\alpha^2)}\right] = \sigma_x^2 \frac{\alpha^2(1-\rho^2)}{1+2\alpha\rho+\alpha^2} \quad (*)$$

where $\alpha = \sigma_v/\sigma_x$. [5]

(ii) Suppose $\alpha \neq 1$

Then there is no real value of α for which $1+2\alpha\rho+\alpha^2=0$.

It is clear that the mean square error $(*)$ is minimized when $\rho=+1$ and $\rho=-1$ (<u>two</u> minimizers)

(iii) Suppose $\alpha=1$ <u>and</u> $\rho=-1$. In this case the formula

$$E[|x-\hat{x}|^2] = \sigma_x^2 \frac{\alpha^2(1-\rho^2)}{1+2\alpha\rho+\alpha^2} \text{ " = " } \frac{0}{0}$$

i.e. it is undeterminate.

Note however that, by (a), $V=-X$, so

$$y = X+V = X-X = 0$$

i.e. the random variable y is the zero vector and provides no information about X. Therefore the mean square estimation error is $E[|x|^2] = \underline{\sigma^2}$ [5]

4 Since all random variables involved have zero mean, the 'constant' component in the linear least squares estimator is zero. By symmetry

$$\hat{x} = \alpha \sum_{i=1}^{N} y_i \quad \text{(for some } \alpha\text{)}$$

The mean square error is

$$J(\alpha) = E[|x - \alpha \textstyle\sum_i y_i|^2] = E[|x - \alpha \sum_{i=1}^{N} (x + e_i)|^2]$$

$$= E[|(1-\alpha N)x - \alpha \textstyle\sum_i e_i|^2]$$

$$= (\alpha N - 1)^2 E[x^2] + \alpha^2 N E[e_i^2]$$

$$= (\alpha N - 1)^2 \sigma_x^2 + \alpha^2 N \sigma_e^2$$

The minimizing value of α, α^*, satisfies

$$2(\alpha^* N - 1)\sigma_x^2 N + 2\alpha^* N \sigma_e^2 = 0$$

Hence $\alpha^* = \dfrac{\sigma_x^2}{N\sigma_x^2 + \sigma_e^2}$.

The linear least squares estimate is therefore

$$\hat{x} = \frac{\sigma_x^2}{N\sigma_x^2 + \sigma_e^2} \sum_{i=1}^{N} y_i$$

The mean square error

$$J(\alpha^*) = \left(\frac{N\sigma_x^2}{N\sigma_x^2 + \sigma_e^2} - 1\right)^2 \sigma_x^2 + \frac{N\sigma_x^4 \sigma_e^2}{(N\sigma_x^2 + \sigma_e^2)^2}$$

$$= \frac{\sigma_e^4 \sigma_x^2 + N\sigma_x^4 \sigma_e^2}{(N\sigma_x^2 + \sigma_e^2)^2} = \frac{\sigma_x^2 \sigma_e^2 (\sigma_e^2 + N\sigma_x^2)}{(N\sigma_x^2 + \sigma_e^2)^2} = \frac{\sigma_x^2 \sigma_e^2}{\sigma_e^2 + N\sigma_x^2}$$

For $\sigma_e^2 = 1 \text{ cm}^2$ and $\sigma_x^2 = 0.5$ cm

$$J(\alpha^*) = \frac{0.5 \times 1}{1 + 0.5N}$$

but $\dfrac{1}{2+n} = 0.01$ when $100 = 2 + n$ or $n = 98$

Since $\dfrac{1}{2+n}$, $n = 1, 2, \ldots$ is decreasing

$$n = 98$$

is the minimum number of sensors, consistent with the constraint.

5 (i) $p_1(y_1,\dots,y_N) = \prod_{k=1}^{N} (2\pi\sigma^2)^{-\frac{1}{2}} \exp\{-\frac{1}{2}\frac{(y-a^k)^2}{\sigma^2}\}$ and

$p_0(y_1,\dots,y_N = \prod_{k=1}^{N} (2\pi\sigma^2)^{-\frac{1}{2}} \exp\{-\frac{1}{2} y^2/\sigma^2\}$

So

$$l = \sum_k (-\tfrac{1}{2})\left(\frac{(y-a^k)^2}{\sigma^2}\right) + \tfrac{1}{2}\, y^2/\sigma^2 = \sum_k (\tfrac{1}{2})\left(\frac{2ya^k}{\sigma^2} - \frac{a^{2k}}{\sigma^2}\right) = \frac{a^k}{2\sigma^2}[2y - a^k]$$

We have shown $\underline{l = \sigma^{-2} \sum_{k=1}^{N} a^k [y_k - \tfrac{1}{2}a^k]}$

(ii) Assume y_k are independent and $y_k \sim N(0,\sigma^2)$

$$l = \sigma^{-2}\sum_{k=1}^{N} a^k y_k - \frac{\sigma^{-2}}{2}\sum a^{2k}$$

Then, $l \sim N\left(-\frac{1}{2\sigma^2}\sum_{k=1}^{N} a^{2k}, \frac{1}{\sigma^2}\sum_{k=1}^{N} a^{ek}\right)$

So

$$\frac{l + \frac{1}{2\sigma^2}\sum_{k=1}^{N} a^{2k}}{\frac{1}{\sigma^2}\sqrt{\sum_{k=1}^{N} a^{2k}}} \sim N(0,1)$$

(iii) The Neyman Pearson test is:

Choose H_1 if $l(y_1,\dots,y_N) > \eta$

Choose H_0 if $l(y_1,\dots,y_N) < \eta$

where η is chosen so that

$$P_0(l > \eta) \doteq 0.01$$

or

$$P_0\left(\underbrace{\frac{l + \frac{1}{2\sigma^2}\sum_k a^{2k}}{\frac{1}{\sigma^2}\sqrt{\sum_k a^{2k}}}}_{(a)} > \frac{\eta + \frac{1}{2\sigma^2}\sum_k a^{2k}}{\frac{1}{\sigma^2}\sqrt{\sum_k a^{2k}}}\right) = 0.01$$

Since (a) has density $N(0,1)$, we require

$$\eta = \alpha \times \left(\frac{1}{\sigma^2}\sqrt{\sum_k a^{2k}}\right) - \frac{1}{2\sigma^2}\sum_k a^{2k}$$

Here α is a constant chosen so that

$$1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-x'/2)\,dx' = 0.01.$$

6(a) The EKF is based on the assumptions:

$$x_k = f(x_{k-1}) + w_k \simeq f(\hat{x}_{k-1}) + f_x(\hat{x}_{k-1})(x_{k-1} - \hat{x}_{k-1}) + w_k, \text{ and}$$

$$y_k = h(x_k) + v_k \simeq h(f(\hat{x}_{k-1})) + h_x(f(\hat{x}_{k-1}))(x_k - f(\hat{x}_{k-1})) + v_k$$

Since $E[x_{k-1} - \hat{x}_{k-1} \mid y_{1:k-1}] = 0$, the standard Kalman filter eqns give

$$P_{k|k-1} = FP_{k-1}F^T + Q^s, \quad P_k = P_{k|k-1} - P_{k|k-1}H^T[HP_{k|k-1}H^T + Q^m]^{-1}HP_{k|k-1}$$

$$K_k = P_{k|k-1}H^T[HP_{k|k-1}H^T + Q^m]^{-1}$$

and $\hat{x}_k = f(\hat{x}_{k-1}) + K_k[y_k - h(f(\hat{x}_{k-1}))]$

where $F = f_x(\hat{x}_{k-1})$ and $H = h_x(f(\hat{x}_{k-1}))$.

For the given special case, $H = \left(\partial h/\partial x_1 (F\hat{x}_{k-1}), \ \partial h/\partial x_2 (F\hat{x}_{k-1})\right)$

But $\partial h/\partial x_1 = \partial/\partial x_1 (x_1^2 + x_2^2)^{1/2} = (x_1^2 + x_2^2)^{-\frac{1}{2}} x_1$. Also $\frac{\partial h}{\partial x_2} = (\ldots)^{-1/2} x_2$

So $H = \|F\hat{x}_{k-1}\|^{-1} \hat{x}_{k-1}^T F^T$

(b) $\Phi_y(\omega) = D(z)|_{z=e^{j\omega}}$, where $D(z) = \dfrac{-2z + 5 - 2z^{-1}}{3z + 10 + 3z^{-1}}$. But

$$D(z) = -\frac{(2z^2 - 5z + 2)}{(3z^2 + 10z + 3)} = \frac{-(2z-1)(z-2)}{(3z+1)(z+3)} = \left(\frac{2}{3}\right)^2 \frac{(1 - \frac{1}{2}z^{-1})}{(1 + \frac{1}{3}z^{-1})} \cdot \frac{(1 - \frac{1}{2}z)}{(1 + \frac{1}{3}z)}$$

The spectral density $\Phi_y(\omega)$ is therefore "realised" by

$(3 + z^{-1})\, y_k = (2 - z^{-1})\, \tilde{e}_k$, in which $\{\tilde{e}_k\}$ is white, with unit variance.

The system in state space form has transfer function

$$c[zI - A]^{-1}b = [c_0 \ c_1]\begin{bmatrix} z & -1 \\ +a_0 & z + a_1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{z^2 + a_1 z + a_0}[c_0 \ c_1]\begin{bmatrix} z + a_1 & +1 \\ -a_0 & z \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{c_0 + c_1 z}{z^2 + a_1 z + a_0} = \frac{z^{-1}(c_1 + c_0 z^{-1})}{(+a_0 z^{-2} + a_1 z^{-1} + 1)}$$

So, if $y_k = c[zI - A]^{-1}b$,

$$3(1 + a_1 z^{-1} + a_0 z^{-2})\, y_k = 3(c_1 + c_0 z^{-1})\, e_{k-1}$$

(white noise, unit variance)

Matching equations gives:

$$3a_1 = 1, \quad a_0 = 0, \quad 3c_1 = 2, \quad 3c_0 = -1$$

Hence

$$a_1 = \frac{1}{3}, \quad a_0 = 0, \quad c_1 = \frac{2}{3}, \quad c_0 = -1/3$$

[12]