Paper Number(s): **E4.13**
AS2
SO15
ISE4.31

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

MSc and EEE/ISE PART IV: M.Eng. and ACGI

SPECTRAL ESTIMATION AND ADAPTIVE SIGNAL PROCESSING

Monday, 21 May 10:00 am

There are FIVE questions on this paper.

Answer THREE questions.

Time allowed: 3:00 hours

Corrected Copy
Q5c @ 11.35am

Examiners: Clark,J.M.C. and Allwright,J.C.

Special instructions for invigilators: None

Information for candidates: None

1.

The mean square error performance function for the N coefficient Finite Impulse Response (FIR) filter represented in Figure 1 is given by

$$J(\underline{w}) = \sigma_d^2 - 2\underline{p}^T\underline{w} + \underline{w}^T R \underline{w}$$

where σ_d^2 is the variance of the desired response $\{d[n]\}$, R is the autocorrelation matrix, $E\{\underline{x}[n]\underline{x}^T[n]\}$, with $\underline{x}[n] = [x[n], x[n-1], \ldots, x[n-N+1]]^T$, $\underline{p}$ is the cross-correlation vector $E\{d[n]\underline{x}[n]\}$ and $\underline{w} = [w_1, w_2, \ldots w_N]^T$ is the vector of coefficients.

(a) State the assumption on the nature of the autocorrelation matrix R so that $J(\underline{w})$ has a unique minimum, and give an example input signal, $\{x[n]\}$, which would satisfy this assumption. **[2 marks]**

(b) Sketch the contours of constant $J(\underline{w})$, as a function of $\underline{w}$, when

(i) $\underline{p} = \begin{bmatrix} 0.9 \\ 0.85 \end{bmatrix}$, $R = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$ and (ii) $\underline{p} = \begin{bmatrix} 0.9 \\ 0.85 \end{bmatrix}$, $R = \begin{bmatrix} 1.0 & 0.9 \\ 0.9 & 1.0 \end{bmatrix}$ **[6 marks]**

(c) Describe how the method of steepest descent, as described by the recursion

$$\underline{w}[k+1] = \underline{w}[k] - \frac{\alpha}{2}\nabla_{\underline{w}} J(\underline{w}[k])$$

may be used to converge in the mean to the minimum of $J(\underline{w})$ **[4 marks]**

(d) For $J(\underline{w})$ in (b) (ii) and given that the autocorrelation matrix R can be replaced by the similarity transform

$$R = Q\Lambda Q^T$$

where Q is the matrix of normalised eigenvectors of R and Λ is the diagonal matrix of corresponding eigenvalues; hence, or otherwise, show that the steepest descent solution may be written as

$$\begin{bmatrix} w_1[k] \\ w_2[k] \end{bmatrix} = \begin{bmatrix} 0.71 - 0.46(1-1.9\alpha)^k - 0.25(1-0.1\alpha)^k \\ 0.21 - 0.46(1-1.9\alpha)^k + 0.25(1-0.1\alpha)^k \end{bmatrix}$$

[13 marks]

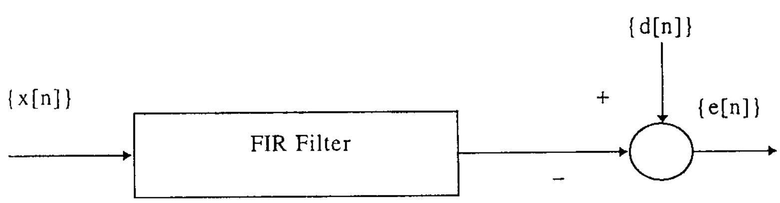


Figure 1

2.

(a) Discuss the relationship between linear prediction and autoregressive modelling.

[5 marks]

(b) For an autoregressive process generated by the difference equation

$$x[n] = \frac{14}{24}x[n-1] - \frac{9}{24}x[n-2] - \frac{1}{24}x[n-3] + w[n]$$

where w[n] is a zero mean statistically stationary white noise discrete time signal with variance σ_w^2

(i) Calculate the coefficients of the optimum linear predictor. **[1 mark]**

(ii) Using the step down algorithm, as described by

$$a_{k-1}[i] = \frac{a_k[i] - a_k[k]a_k[k-i]}{1 - a_k^2[k]} \quad i = 1,2,\ldots,k-1;\ k = p,p-1,\ldots,2$$

evaluate the reflection coefficients which correspond to the optimum linear predictor and show how these can be used in a lattice structure realisation. **[6 marks]**

(iii) Determine the autocorrelation sequence $r_{xx}[\tau]$ for $|\tau| \leq 2$.

[6 marks]

(c) Describe the steps involved in the autocorrelation method for power spectrum estimation from an observation $\{x[0],x[1],\ldots,x[N-1]\}$, commenting upon the computational complexity of each step.

[7 marks]

3.

(a) Describe where moving average models are used to represent elements of mobile communication systems.

[3 marks]

(b) A zero mean, white noise process, $\{w[n]\}$, with variance σ_w^2 is input to a moving average MA(q) filter with impulse response sequence $\{b[0],b[1],...,b[q]\}$, calculate the mean value of the output of such a filter and verify that its autocorrelation function is given by

$$r_{MA}[\tau] = \sigma_w^2 \sum_{m=0}^{q-|\tau|} b[m]b[m+\tau]$$

[7 marks]

(c) The autocorrelation sequence of an MA(2) process is found to be $r_{MA}[0] = 6\sigma_w^2$, $r_{MA}[\pm1] = -4\sigma_w^2$ and $r_{MA}[\pm2] = 2\sigma_w^2$, and is otherwise zero.

(i) Evaluate the impulse response sequence of the MA(2) filter.

(ii) State whether the solution in (i) is unique and, if there is more than one solution, describe the different solutions and why they exist.

[10 marks]

(d) If the output of a MA model is corrupted by additive coloured Gaussian measurement noise, suggest a method to estimate the impulse response of the model which is immune to such noise, and state any assumptions that are necessary.

[5 marks]

4.

A set of linear equations is represented in matrix form by

$$A\underline{x} = \underline{b}$$

where A is an n x m matrix with known complex elements, $\underline{x}$ is an m-dimensional vector, the elements of which are the unknowns, and $\underline{b}$ is an n-dimension vector with known complex elements.

(a) Show, for the three cases n = m, n > m, and n< m, the form of the solution for $\underline{x}$, and the corresponding value of the cost function $J = \underline{e}^H \underline{e}$, where $\underline{e} = \underline{b} - A\underline{x}$, and $(.)^H$ denotes Hermitian transpose.

[8 marks]

(b) A communications array measurement signal is modelled in the form

$$m[k] = \mu + \alpha \exp(j2\pi f_o k) \; ; \; k = 0,1,\ldots,N-1$$

where μ is a complex d.c. level and α is the amplitude of the complex sinusoid. Formulate the solution for μ and α as a set of overdetermined equations and show that the least squares solution for μ and α is given by

$$\begin{bmatrix} \mu \\ \alpha \end{bmatrix} = \begin{bmatrix} \dfrac{N\,DFT(0) - \exp(j\pi f_o[N-1])S(f_o)DFT(f_o)}{N^2 - S^2(f_o)} \\ \dfrac{N\,DFT(f_o) - \exp(-j\pi f_o[N-1])S(f_o)DFT(0)}{N^2 - S^2(f_o)} \end{bmatrix}$$

where $S(f_o) = \dfrac{\sin \pi f_o N}{\sin \pi f_o}$ and $DFT(f) = \sum_{k=0}^{N-1} m[k] \exp(-j2\pi f k)$.

[10 marks]

(c) Show how the least squares solution in (b) simplifies when N is even and $f_o = \dfrac{p}{N}$ where p is a nonzero integer in the range $\left[-\dfrac{N}{2}, \dfrac{N}{2} - 1\right]$.
Comment upon the result.

[4 marks]

(d) As the frequency f_o of the model is generally unknown suggest methods by which this may be estimated. **[3 marks]**

5.

A family of stochastic gradient algorithms is based upon approximately minimising cost functions of the form

$$J = E\{e^{2p}[n]\} \quad p = 1,2,3,...$$

where $e[n] = d[n] - \hat{d}[n]$, namely the difference between the desired response $d[n]$ and the output of the adaptive filter $\hat{d}[n] = \underline{w}^T[n]\underline{x}[n]$, where $\underline{w}[n] = [w_1[n], w_2[n], ..., w_N[n]]^T$ is the coefficient vector of an N-tap, finite impulse response, adaptive filter with input vector $\underline{x}[n] = [x[n], x[n-1], ..., x[n-N+1]]^T$.

(a) Explain when it would be advantageous to use an adaptive algorithm based on $p \geq 2$ and give an example application.

[3 marks]

(b) Verify that a least mean square (LMS) type coefficient update for $\underline{w}[n]$, based upon J, is given by

$$\underline{w}[n+1] = \underline{w}[n] + 2p\mu e^{2p-1}[n]\underline{x}[n]$$

[3 marks]

(c) Given that $d[n] = \underline{w}^T\underline{x}[n] + v[n]$ where $\underline{w} = [w_1, w_2, ..., w_N]^T$ is a vector of fixed, but unknown parameters, and $v[n]$ is zero mean independent identically distributed white noise with symmetric probability density function, which is statistically independent of $\underline{x}[n]$, and that the weight error vector, $\underline{c}[n] = \underline{w}[n] - \underline{w}$, is close to zero, show that

2

$$E\{\underline{c}[n+1]\} = [I - \mu p(2p-1)E\{v^{2p-2}[n]\}R]E\{\underline{c}[n]\}$$

X

[12 marks]

(d) Establish the conditions on the adaptation gain, μ, that assures that the mean $E\{\underline{w}[n]\}$ of the coefficient vector of the adaptive filter converges to the desired vector $\underline{w}$.

[7 marks]

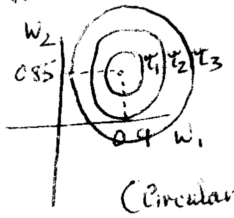
E4.13, ISE4.31   2001   (SEASP) (Feb 01)

AS2, SC15

(1) (a) Positive definite, $\underline{x}^T R \underline{x} > 0 \quad \forall \underline{x} \neq \underline{0}$ (2)

white noise, or N sinusoids with different frequencies

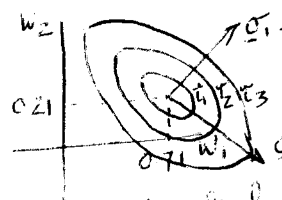
(b) (i) $\underline{w}_{opt} = R^{-1}\underline{p} = \begin{bmatrix} 0.9 \\ 0.85 \end{bmatrix}$



$J = \tau_1, \tau_2$ or τ_3 with $\tau_1 < \tau_2 < \tau_3$

(Circular contours)

(ii) $\underline{w}_{opt} = R^{-1}\underline{p} \doteq \begin{bmatrix} 0.71 \\ 0.21 \end{bmatrix}$



$\underline{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (6)

(Elliptical contours)

(c) Begin with $\underline{w}(0) = \underline{0}$, iterate for $k = 1, 2, \ldots$

Select α to satisfy $0 < \alpha < \frac{1}{\lambda_{MAX}}$

$\nabla_{\underline{w}} J(\underline{w}) = -2\underline{p} + 2R\underline{w}$, $\quad \underline{w}[k+1] = \underline{w}[k] + \alpha(\underline{p} - R\underline{w}[k])$

$\lim_{k \to \infty} \underline{w}[k+1] \to \underline{w}_{opt} = R^{-1}\underline{p}$ (4)

(d) $\underline{w}[k+1] = \underline{w}[k] + \alpha(R\underline{w}_{opt} - R\underline{w}[k])$

$\underline{w}[k+1] - \underline{w}_{opt} = (\underline{w}[k] - \underline{w}_{opt}) - \alpha R(\underline{w}[k] - \underline{w}_{opt})$, $\quad \underline{v}[k] \triangleq (\underline{w}[k] - \underline{w}_{opt})$

$\underline{v}[k+1] = \underline{v}[k] - \alpha R \underline{v}[k] = (I - \alpha R)\underline{v}[k]$ — (I)

$R = Q\Lambda Q^T$ — eigenvalues found from $|R - \lambda I| = 0$

So, from characteristic equation $\lambda_1 = 1.9$, with $\underline{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as in (b)(ii),

$\lambda_2 = 0.1$, with $\underline{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

and $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\Lambda = \text{Diag}(1.9, 0.1)$

From (I) and using $Q^TQ = I$

$Q^T\underline{v}[k+1] = (I - \alpha\Lambda)Q^T\underline{v}[k]$, $\quad \underline{v}'[k] = Q^T\underline{v}[k]$,

thus $\underline{v}'[k+1] = (I - \alpha\Lambda)\underline{v}'[k]$, by induction $\underline{v}'[k] = (I - \alpha\Lambda)^k \underline{v}'[0]$

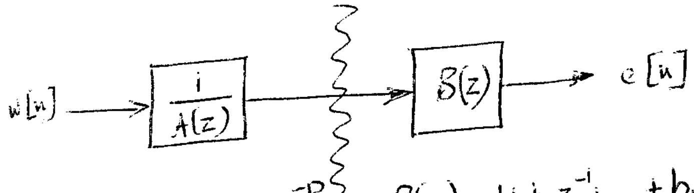
with $\underline{v}'[0] = Q^T\underline{v}[0] = \frac{1}{\sqrt{2}}\begin{bmatrix} -0.92 \\ -0.5 \end{bmatrix}$

$\underline{v}[k] = Q\underline{v}'[k] = -\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} 0.92(1-\alpha 1.9)^k \\ 0.5(1-\alpha 0.1)^k \end{pmatrix}$ (13)

and $\underline{w}[k] = \underline{v}[k] + \underline{w}_{opt} = \begin{bmatrix} 0.71 - 0.46(1-\alpha 1.9)^k - 0.25(1-\alpha 0.1)^k \\ 0.21 - 0.46(1-\alpha 1.9)^k + 0.25(1-\alpha 0.1)^k \end{bmatrix}$

25/25

(2) (a) AR Model Predictor (Linear forward prediction error filter)



$A(z) = 1 + a_1 z^{-1} + \dots + a_p z^{-p}$ $B(z) = 1 + b_1 z^{-1} + \dots + b_p z^{-p}$

$b_i;\ i = 1, 2, \dots, p$ designed to minimize $E\{e^2[n]\}$,

then $b_i = a_i\ \forall i$ and hence $e[n] = w[n]$. (5)

(b) (i) $B(z) = 1 + a_3[1]z^{-1} + a_3[2]z^{-2} + a_3[3]z^{-3}$ (1)

$a_3[1] = \frac{-14}{24}$, $a_3[2] = \frac{9}{24}$, $a_3[3] = \frac{1}{24}$

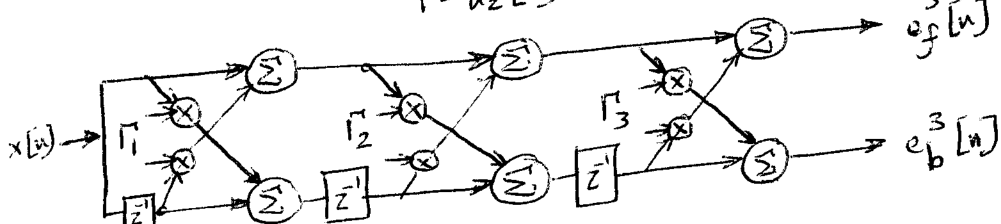
(ii) $a_3[3] = \frac{1}{24} = \Gamma_3$

Using the step down algorithm

$k=3,\ i=1$ $\quad a_2[1] = \dfrac{a_3[1] - a_3[3]a_3[2]}{1 - a_3^2[3]} = \dfrac{-\frac{14}{24} - \frac{1}{24} \times \frac{9}{24}}{1 - \frac{1}{24 \times 24}} = -0.6$

$k=3,\ i=2$ $\quad a_2[2] = \dfrac{a_3[2] - a_3[3]a_3[1]}{1 - a_3^2[3]} = \dfrac{\frac{9}{24} + \frac{1}{24} \times \frac{14}{24}}{1 - \frac{1}{24 \times 24}} = 0.4 = \Gamma_2$

$k=2,\ i=1$ $\quad a_1[1] = \dfrac{a_2[1] - a_2[2]a_2[1]}{1 - a_2^2[2]} = \dfrac{-0.6 + 0.4 \times 0.6}{1 - 0.4^2} = -\frac{3}{7} = \Gamma_1$



(6)

(iii) $r_{xx}(0) = \dfrac{\sigma_w^2}{\prod_{i=1}^{3}(1 - \Gamma_i^2)} = 1.456\sigma_w^2$

$r_{xx}(-1) = r_{xx}(1) = -a_1(1)r_{xx}(0) = \frac{3}{7} \times 1.917\sigma_w^2 = 0.625\sigma_w^2$

$\rho_1 = r_{xx}(0)(1 - \Gamma_1^2) = 1.193\sigma_w^2$ (6)

$r_{xx}(-2) = r_{xx}(2) = -a_1(1)r_{xx}(1) - a_2(2)\rho_1 = -0.208\sigma_w^2$

(c) Step 1

ACF Estimation

$\hat{r}_{xx}(k) = \frac{1}{N}\sum_{n=0}^{N-1-k} x[n]x[n+k],\ k = 0, 1, \dots, p$

$= \hat{r}_{xx}(-k)$ $\sim O(N(p+1))$

Step 2

Solve normal eqns with Lev. algo

$$\begin{bmatrix} \hat{r}_{xx}[0] & \cdots & \hat{r}_{xx}[p-1] \\ \vdots & & \vdots \\ \hat{r}_{xx}[p-1] & \cdots & \hat{r}_{xx}[0] \end{bmatrix} \begin{bmatrix} \hat{a}[1] \\ \vdots \\ \hat{a}[p] \end{bmatrix} = -\begin{bmatrix} \hat{r}_{xx}(1) \\ \vdots \\ \hat{r}_{xx}(p) \end{bmatrix} \sim O(p^2)$$

$\hat{\sigma}^2 = \hat{r}_{xx}(0) + \sum_{k=1}^{p} \hat{a}[k]\hat{r}_{xx}(-k)$ $\sim O(p)$

Step 3 (7)

$\hat{P}_{xx}(f) = \dfrac{\hat{\sigma}^2}{|1 + \hat{a}[1]e^{-j2\pi f} + \dots + \hat{a}[p]e^{-j2\pi pf}|^2}$

Use FFT

$O(N \log_2 N)$

25/25

3) a) Multipath channel modelling as in MLSE equalizer, as in GSM
Inter loudspeaker, microphone, impulse response, as in acoustic echo cancellation (3)

b) $y[k] = \sum_{m=0}^{q} b(m) w[k-m]$

$E\{w[k]\} = 0$, $E\{w[k] w[k+\tau]\} = \sigma_w^2 \delta[\tau]$

$E\{y[k]\} = \sum_{m=0}^{q} b(m) E\{\cancel{w[k-m]}\}^{0} = 0$

ACF

$r_{MA}(\tau) = E\{y[k] y[k+\tau]\} = \sum_{m=0}^{q} \sum_{s=0}^{q} b(m) b(s) \underbrace{E\{w[k-m] w[k+\tau-s]\}}_{\sigma_w^2 \delta(s-[\tau+m])}$

$= \sigma_w^2 \sum_{m=0}^{q} b(m) b(m+\tau)$ Since $b(m) = 0 \ \forall \ |m| > q$

$= \sigma_w^2 \sum_{m=0}^{q-|\tau|} b(m) b(m+\tau)$ (7)

c) From $r_{MA}(\tau)$

$\sigma_w^2 (b^2(0) + b^2(1) + b^2(2)) = 6\sigma_w^2$ - I

$\sigma_w^2 (b(1)[b(0) + b(2)]) = -4\sigma_w^2$ - II

$\sigma_w^2 (b(0) b(2)) = 2\sigma_w^2$ - III

Using II + III $b(1) = \dfrac{-4b(0)}{b^2(0)+2}$ $b(0) = \dfrac{2}{b(2)}$

With $s \triangleq b^2(0)$, from I $(s-2)(s^3-s) = s^4 - 2s^3 - 8s + 16 = (s-2)^2(s^2+2s+4) = 0$

From the real root, $b^2(0) = 2 \Rightarrow b(0) = \pm\sqrt{2}$

(i) $b(0) = \sqrt{2}$, $b(1) = -\sqrt{2}$, $b(2) = \sqrt{2}$

(ii) not unique, $b(0) = -\sqrt{2}$, $b(1) = \sqrt{2}$, $b(2) = -\sqrt{2}$ another solution;
ACF is symmetric, no phase information (10)

d) Employ higher order statistics, assume input to AR model is third order white

$y(n) = x_{MA}(n) + w_G(n)$

$r_y(\tau_1, \tau_2) = r_{x_{MA}}(\tau_1, \tau_2)$ since $r_{w_G}(\tau_1, \tau_2) = 0 \quad \forall \ \tau_1, \tau_2$

$= \gamma_3 \sum_{m=0}^{q-\max(|\tau_1|,|\tau_2|)} b(m) b(m+\tau_1) b(m+\tau_2)$

Use cumulant matching to solve for $b(i)$ (5)

i.e. $\min_{b(i)} \left| \hat{r}_y(\tau_1, \tau_2) - \gamma_3 \sum_{m=0}^{q-\max(|\tau_1|,|\tau_2|)} b(m) b(m+\tau_1) b(m+\tau_2) \right|^2$

$\tau_1, \tau_2 \in R_{EA}$

25/25

4)

a) $n = m$ exactly determined

$x = A^{-1}\underline{b}$ - provided rank$(A) = n$, $J = 0$.

$n > m$ - overdetermined

$\underline{x} = (A^H A)^{-1} A^H \underline{b}$, $J = (\underline{b} - A\underline{x})^H(\underline{b} - A\underline{x})\Big|_{\underline{x} = (A^H A)^{-1} A^H \underline{b}}$

$= \underline{b}^H \underline{b} - \underline{b}^H A (A^H A)^{-1} A^H \underline{b}$ since $A^H \underline{e} = \underline{0}$

$= \underline{b}^H P^{\perp} \underline{b}$ where $P^{\perp} = (I - A(A^H A)^{-1} A^H)$

$n < m$ - underdetermined

One approach, $\min \|\underline{x}\|_2$ s.t. $A\underline{x} = \underline{b}$

$x = A^H (A A^H)^{-1} \underline{b}$ where $A^H (A A^H)^{-1}$ - pseudo inverse, $J = 0$. (8)

b)
$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & \exp(j2\pi f_o) \\ \vdots & \vdots \\ 1 & \exp(j2\pi f_o[N-1]) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \mu \\ \alpha \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} m[0] \\ m[1] \\ \vdots \\ m[N-1] \end{bmatrix}}_{\underline{b}}$$

c) $N > 2$,
$$\begin{bmatrix} \mu \\ \alpha \end{bmatrix}_{LS} = (A^H A)^{-1} A^H \underline{b}, \quad A^H A = \begin{bmatrix} N & \sum_{k=0}^{N-1} \exp(j2\pi f_o k) \\ \sum_{k=0}^{N-1} \exp(-j2\pi f_o k) & N \end{bmatrix},$$

$$\sum_{k=0}^{N-1} \exp(j2\pi f_o k) = \exp(j2\pi f_o[N-1]) \underbrace{\frac{\sin \pi f_o N}{\sin \pi f_o}}_{S(f_o)}, \text{ thus } (A^H A)^{-1} = \frac{1}{N^2 - S^2(f_o)} \begin{bmatrix} N & -\exp(j\pi f_o[N-1])S(f_o) \\ -\exp(-j\pi f_o[N-1])S(f_o) & N \end{bmatrix}$$

$$A^H \underline{b} = \begin{bmatrix} \sum_{k=0}^{N-1} m[k] \\ \sum_{k=0}^{N-1} m[k]\exp(-j2\pi f_o k) \end{bmatrix} = \begin{bmatrix} DFT(0) \\ DFT(f_o) \end{bmatrix}, \quad \begin{bmatrix} \mu \\ \alpha \end{bmatrix}_{LS} = \frac{1}{N^2 - S^2(f_o)} \begin{bmatrix} N\,DFT(0) - \exp(j\pi f_o[N-1])S(f_o)DFT(f_o) \\ N\,DFT(f_o) - \exp(-j\pi f_o[N-1])S(f_o)DFT(0) \end{bmatrix}$$
(10)

c) $f_o = \frac{p}{N}$ $S(f_o) = \frac{\sin \pi p}{\sin \pi f} = 0$, p non zero integer

$$\Rightarrow \begin{bmatrix} \mu \\ \alpha \end{bmatrix}_{L.S} = \begin{bmatrix} \frac{1}{N}\sum_{k=0}^{N-1} m[k] \\ \frac{1}{N}\sum_{k=0}^{N-1} m[k]\exp(-j\frac{2\pi pk}{N}) \end{bmatrix} = \begin{bmatrix} \text{Sample mean} \\ \text{-------} \\ \text{Sample mean of input shifted to } f = p/N \end{bmatrix}$$
(4)

d) many possibilities - non linear least squares, peak detection of FFT following mean removal (3)

25/25

5 a) In system identification if the measurement noise has a sub-Gaussian probability density function, $p=2$ would yield lower misadjustment than $p=1$ (3)

b) LMS-based minimization of instantaneous error squared with an equation of the form -

$$\underline{w}[n+1] = \underline{w}[n] - \mu \nabla_{\underline{w}} \hat{J}\Big|_{\underline{w}=\underline{w}[n]}$$

$$\hat{J} = e^{2p}[n] \quad \nabla_{\underline{w}} \hat{J} = 2pe^{2p-1}[n]\nabla_{\underline{w}} e[n]$$

$$e[n] = d[n] - \underline{w}^T[n]\underline{x}[n], \quad \nabla_{\underline{w}}\hat{J} = -2pe^{2p-1}[n]\underline{x}[n], \quad \underline{w}[n+1] = \underline{w}[n] + 2\mu p e^{2p-1}[n]\underline{x}[n]$$ (3)

c)
$$e[n] = d[n] - \underline{w}^T[n]\underline{x}[n]$$
$$= (\underline{w} - \underline{w}[n])^T \underline{x}[n] + v[n] = -\underline{c}^T[n]\underline{x}[n] + v[n]$$

$$e^{2p-1}[n] = (-\underline{c}^T[n]\underline{x}[n] + v[n])^{2p-1}$$

Since $\underline{c}[n] \approx \underline{0}$

$$e^{2p-1}[n] \approx -(2p-1)\underline{x}^T[n]\underline{c}[n]v^{2p-2}[n] + v^{2p-1}[n]$$

Thus, from update equation in b)

$$\underline{w}[n+1] - \underline{w} = \underline{w}[n] - \underline{w} + 2\mu p e^{2p-1}[n]\underline{x}[n]$$

$$\Rightarrow \underline{c}[n+1] \cong \underline{c}[n] - 2\mu p(2p-1)v^{2p-2}[n]\underline{x}[n]\underline{x}^T[n]\underline{c}[n]$$
$$+ 2\mu p v^{2p-1}[n]\underline{x}[n]$$

Taking $E\{\ \}$

$$E\{\underline{c}[n+1]\} \cong (I - 2\mu p(2p-1)E\{v^{2p-2}[n]\}R_{xx})E\{\underline{c}[n]\}$$ (12)

d) $R = Q\Lambda Q^T$, thus

$$E\{\underline{c}[n+1]\} \cong (I - 2\mu p(2p-1)E\{v^{2p-2}[n]\}Q\Lambda Q^T)E\{\underline{c}[n]\}$$

$$E\{\underline{c}[n+1]\} \cong (I - 2\mu p(2p-1)E\{v^{2p-2}[n]\}\Lambda)E\{Q^T\underline{c}[n]\}, \text{ where } \Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_N)$$

$$\Rightarrow E\{Q^T\underline{c}[n+1]\} \cong (I - 2\mu p(2p-1)E\{v^{2p-2}[n]\}\Lambda)E\{Q^T\underline{c}[n]\}$$

For convergence of all modes

$$|1 - 2\mu p(2p-1)E\{v^{2p-2}[n]\}\lambda_i| < 1 \quad \forall i$$

$$|1 - 2\mu p(2p-1)E\{v^{2(p-1)}[n]\}\lambda_i < 1$$

$$\Rightarrow -1 < 1 - 2\mu p(2p-1)E\{v^{2(p-1)}[n]\}\lambda_i$$

$$\Rightarrow 0 < \mu < \frac{1}{p(2p-1)E\{v^{2(p-1)}[n]\}\lambda_i}$$ (7)

Worst case $\lambda_i = \lambda_1 = \lambda_{MAX}$

Jonathan Chambers
29-01-01

25/25