Probability and Stochastic Processes 2003 Model Answers

1 (a) Let A be the event 'there is a closed path'. We have

$$A = A_4 \cup ((A_1 \cup A_2) \cap A_3)$$

So
$$P(\bar{A}) = P(\bar{A}_4 \cap (\overline{(A_1 \cup A_2) \cap A_3}))$$
$$= P(\bar{A}_4) \cdot P(\overline{(A_1 \cup A_2) \cap A_3}) \quad \text{(by independence)}$$
$$= P(\bar{A}_4) \cdot [1 - P((A_1 \cup A_2) \cap A_3)]$$
$$= P(\bar{A}_4) \cdot [1 - P(A_1 \cup A_2) \cdot P(A_3)] \quad \text{(by independence)}$$
$$= P(\bar{A}_4) \cdot [1 - (1 - P(\bar{A}_1 \cap \bar{A}_2)) \cdot P(A_3)]$$
$$= P(\bar{A}_4) \cdot [1 - (1 - P(\bar{A}_1) \cdot P(\bar{A}_2)) \cdot P(A_3)] \quad \text{(by independence)}$$
$$= (1-p)\,[1 - (1 - (1-p)^2)\,p]$$
$$= (1-p)\,[(1-p) + (1-p)^2 p] = (1-p)^2(1 + (1-p)p)$$

Hence $P(A)\ (= 1 - P(\bar{A})) = \underline{1 - (1-p)^2(1 + (1-p)p)}$.

(b) We want

$P(A_1 | B_1)$ (the probability that the source is S_1 when the receiver indicates 'S_1')

By Bayes Rule

$$P(A_1 | B_1) = P(B_1 | A_1)\, P(A_1) / P(B_1). \qquad — (*)$$

But

A_1, A_2, A_3 are disjoint events and

$$A_1 \cup A_2 \cup A_3 = \Omega \quad (\text{'certain event'})$$

It follows $B_1 \cap A_1$, $B_1 \cap A_2$, $B_1 \cap A_3$ are disjoint events and

$$B_1 = (B_1 \cap A_1) \cup (B_1 \cap A_2) \cup (B_1 \cap A_3)$$

Hence
$$P(B_1) = P(B_1 | A_1)P(A_1) + P(B_1 | A_2)P(A_2) + P(B_1 | A_3)P(A_3)$$
$$= 0.8 \times 0.9 + 0.1 \times 0.05 + 0.1 \times 0.05$$
$$= 0.73$$

But then, from (*),

$$P(A_1 | B_1) = \frac{0.8 \times 0.9}{0.73} = \underline{0.986}$$

2 $F_{X/Y}(x|Y=0) = P[X(\omega) \le x \text{ and } 0 \le X(\omega) \le 1] \,/\, P[0 \le X(\omega) \le 1]$

$= \frac{1}{2}\int_0^{\min\{x,1\}} dx \,/\, \int_0^1 \frac{1}{2}\, dx = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 \le x \\ 0 & \text{otherwise} \end{cases}$

Also

$F_{X/Y}(x|Y=1) = P[X(\omega) \le x \text{ and } 1 \le X(\omega)] \,/\, P[X(\omega) \ge 1]$

$= \frac{1}{2}\int_1^{\max\{1,x\}} dx \,/\, \int_1^2 \frac{1}{2}\, dx = \begin{cases} x-1 & \text{if } 1 \le x \le 2 \\ 1 & \text{if } 2 \le x \\ 0 & \text{otherwise} \end{cases}$

These distributions have densities

$f_{X/Y}(x|Y=0) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}, \quad f_{X/Y}(x|Y=1) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$

The conditional mean is

$E[X|Y=0] = \int_0^1 x\, dx = \frac{1}{2}$ and $E[X|Y=1] = \int_1^2 x\, dx = 1\frac{1}{2}$

The error variance is

$\int_{-\infty}^{+\infty} |X - \hat{X}|^2 dX = \int_0^\infty |x - \hat{x}(0)|^2 P[Y=0]\, dx + \int_0^\infty |x - \hat{x}(1)|^2 P[Y=1]\, dx$

However $P[Y=0] = P[0 \le X(\omega) \le 1] = \int_0^1 \frac{1}{2} dx = \frac{1}{2}$

and $P[Y=1] = P[1 \le X(\omega) \le 2] = \int_0^1 \frac{1}{2} dx = \frac{1}{2}$

Hence

$\int_{-\infty}^{\infty} |x - \hat{x}|^2 dx = \frac{1}{2}\int_0^1 (x - \frac{1}{2})^2 dx + \frac{1}{2}\int_1^2 |x - \frac{3}{2}|^2 dx$

$= \frac{1}{2}\int_{-\frac{1}{2}}^{+\frac{1}{2}} x^2 dx + \frac{1}{2}\int_{-1/2}^{+1/2} x^2 dx = \int_{-1/2}^{+1/2} x^2 dx = \frac{1}{3} x^3 \Big|_{-1/2}^{+1/2}$

$= \frac{2}{3} \cdot \frac{1}{8} = \frac{1}{12}$

We have shown $E[|X(\omega) - \hat{X}(Y(\omega))|^2] = \frac{1}{12}$

The estimator $\hat{X}(y) = \begin{cases} \frac{1}{2} & \text{if } y = 0 \\ 1\frac{1}{2} & \text{if } y = 1 \end{cases}$

This can be expressed as a linear estimator

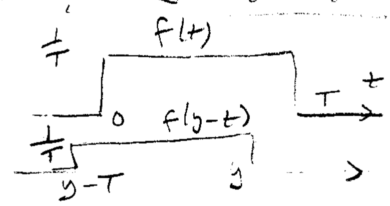
$\hat{x}(y) = y + 0.5$

3 (a) Write $Y(\omega) = T_1(\omega) + T_2(\omega)$. Then

$$f_Y(y)\delta_y = P[y \le T_1(\omega) + T_2(\omega) \le y + \delta_y]$$
$$\simeq \sum_i P[y - t_i \le T_1(\omega) \le y - t_i + \delta_y \text{ and } t_i \le T_2(\omega) \le t_i + \Delta t]$$
$$= \sum_i P[y - t_i \le T_1(\omega) \le y - t_i + \delta_y] \cdot P[t_i \le T_2(\omega) \le t_i + \Delta t] \quad (\Delta t = t_{i+1} - t_i)$$
$$= \int f_{T_1}(y-t) f_{T_2}(t)\, dt \cdot \delta_y$$

Hence $f_Y(y) = \int_{-\infty}^{+\infty} f(y-t) f(t)\, dt$

Evaluating the integral, with the help of the diagram, we see



$$f_Y(y) = \begin{cases} \frac{1}{T^2}\, y & 0 \le y \le T \\ \frac{1}{T^2}(2T - y) & T \le y \le 2T \end{cases}$$

and $f_Y(y) = 0$ outside these ranges.

[10]

(b) $P[Z(\omega) \le z] = P[Z(\omega) \le z \text{ and } F] + P[Z(\omega) \le z \text{ and } \bar{F}]$

$$= P[T_1(\omega) \le z \text{ and } F] + P[T_1(\omega) + T_2(\omega) \le z \text{ and } \bar{F}]$$
$$= P[T_1(\omega) \le z] \cdot P[F] + P[T_1(\omega) + T_2(\omega) \le z] \cdot P[\bar{F}]$$

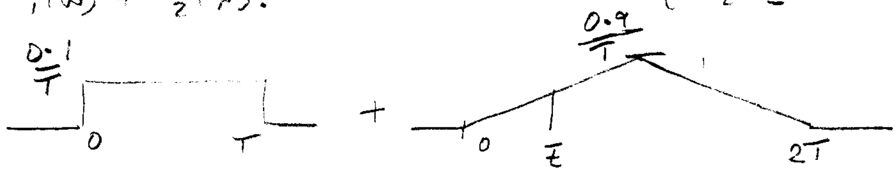
(by independence)

Hence $f_Z(z) = f_{T_1}(z) \cdot P[F] + f_Y(z)\,(1 - P[F])$

where $Y(\omega) = T_1(\omega) + T_2(\omega)$. $\quad (P[F] = 0.1)$

We see that

$f_Z(z) =$



$$= \begin{cases} \frac{0.1}{T} + \frac{0.9}{T^2} z & 0 \le z \le T \\ \frac{0.9}{T^2}(2T - z) & T < z \le 2T \\ 0 & \text{otherwise} \end{cases}$$

[4]

For $\bar{t}$ such that $0 \le \bar{t} < T$

$$P[Z(\omega) \ge \bar{t}] = 1 - \frac{0.1}{T} \times \bar{t} - \frac{1}{2}\bar{t} \times \frac{\bar{t}}{T} \times \frac{0.9}{T}$$
$$= 1 - 0.1(\bar{t}/T) - 0.45(\bar{t}/T)^2$$

We require $P[Z(\omega) \ge \bar{t}] = 0.95$ ('smallest' T case)

Solving $0.95 = 1 - 0.1(\bar{t}/T) - 0.45(\bar{t}/T)^2$ gives $\frac{\bar{t}}{T} = \frac{\sqrt{10} - 1}{9}$

But $\bar{t} = 10$ hrs, so

[6] $$T = \frac{9}{\sqrt{10} - 1} \times 10 = 41.62 \text{ hrs}$$

4 (a) $V(d) = E|D(\omega) - d|^2 = E[D^2(\omega)] - 2d\,E[D(\omega)] + d^2$

[2] $V'(d) = 0$ gives $-2E[D(\omega)] + 2d = 0$ hence minimizing $d = E[D(\omega)]$.

(b) A linear estimator has the form $\hat{X} = aY + bZ + c$. We must choose a, b, c to minimize the least squares criterion

$$J(a,b,c) = E[\,|X(\omega) - aY(\omega) - bZ(\omega) - c|^2]$$

But, by (a), the minimizing c (for given a and b) is

$$c(a,b) = m_X - a m_Y - b m_Z = E[\tfrac{1}{2}B + \tfrac{1}{2}F] - aL - bL = \tfrac{1}{2}L - aL = (\tfrac{1}{2} - a) \times L$$

Then $J(a,b,c(a,b)) = \tilde{J}(a,b) = E[\,|X' - aY' - bZ'|^2]$, where $X' = X - m_X$, etc.

$$\tilde{J}(a,b) = E X'^2 + a^2 E Y'^2 + b^2 E Z'^2$$
$$- 2a E(X'Y') - 2b E(X'Z') + 2ab E[Y'Z']$$

Setting gradients to zero ($\frac{\partial}{\partial a}(\,) = 0$, $\frac{\partial}{\partial b}(\,) = 0$) to find minimum gives

$$E Y'^2 a^* - E(X'Y') + b^* E(Y'Z') = 0$$
$$E Z'^2 b^* - E(X'Z') + a^* E(Y'Z') = 0$$

But

$E Y'^2 = \sigma^2$, $E(X'Y') = E(\frac{1}{2}(F' + B')Y') = \frac{1}{2}\alpha^2$, $E(X'Z') = \ldots = \frac{1}{2}\alpha^2$

$E(Y'Z') = r\sigma^2$. Hence

$$\sigma^2 a^* + r\sigma^2 b^* = \tfrac{1}{2}\alpha^2 \quad \text{and} \quad \sigma^2 b^* + r\sigma^2 a^* = \tfrac{1}{2}\alpha^2$$

By symmetry, $a^* = b^*$. It follows

$$\sigma^2(1+r)a^* = \tfrac{1}{2}\alpha^2. \quad \text{Hence} \quad a^* = b^* = \frac{\alpha^2}{2\sigma^2(1+r)}$$

But then

$$c^* = c(a^*, b^*) = (\tfrac{1}{2} - a^*) \times L$$

Summary: $\hat{X} = a^* Y + b^* Z$, where

[16] $a^* = b^* = \alpha^2 / \{2\sigma^2(1+r)\}$ and $c^* = (\frac{1}{2} - a^*) \times L$.

It is expected that the error variance is least when $r = 0$. In this case the two measurements are uncorrelated and supply the most 'information' about $X(\omega)$. By contrast, if $r = 1$ (to take a different extreme) $Y(\omega)$ and $Z(\omega)$ are linearly related, and knowing $Z(\omega)$

[2] does not add to the knowledge of knowing $Y(\omega)$, for example.

5(a) $x_{k+1} = Ax_k + be_k$ — (1)

Post multiply right side by x_{k+1}^T and left side by $(Ax_k + e_k)^T$ $(= x_{k+1}^T)$:

$$x_{k+1}x_{k+1}^T = Ax_k x_k^T A^T + Ax_k e_k^T b^T + be_k x_k^T A^T + be_k e_k^T b^T \quad —(*)$$

But x_k is a 'linear combination' of $e_{k-1}, e_{k-2}, \ldots$ Since the e_k's are zero mean, uncorrelated, it follows $A E\{x_k e_k^T\} + E\{e_k x_k^T\} A^T = 0$.

Taking expectations across $(*)$ gives

$$E\{x_{k+1}x_{k+1}^T\} = A E\{x_k x_k^T\} A^T + \overset{0}{A E\{x_k e_k^T\} b^T} + \overset{0}{bE\{e_k x_k^T\} A^T} + bE\{e_k e_k^T\} b^T$$

[8] Hence $R_x(0) = A R_x(0) A^T + \sigma^2 bb^T$ — Lyapunov equation

(b) The coupled process can be expressed in terms of $\underline{x} = \begin{pmatrix} y_k \\ w_k \end{pmatrix}$ as

$$\underline{x}_{k+1} = \begin{bmatrix} 0.5 & \alpha \\ 0 & 0.2 \end{bmatrix} \underline{x}_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e_k$$

Write $R_{\underline{x}}(0) = \begin{bmatrix} r_{00} & r_{01} \\ r_{01} & r_{11} \end{bmatrix}$. Then the Lyapunov equation is

$$\begin{bmatrix} r_{00} & r_{01} \\ r_{01} & r_{11} \end{bmatrix} = \underbrace{\begin{bmatrix} 0.5 & \alpha \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} r_{00} & r_{01} \\ r_{01} & r_{11} \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ \alpha & 0.2 \end{bmatrix}} + \sigma^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & \alpha \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} 0.5r_{00} + \alpha r_{01} & 0.2r_{01} \\ 0.5r_{01} + \alpha r_{11} & 0.2r_{11} \end{bmatrix} = \begin{bmatrix} 0.25r_{00} + \alpha r_{01} + \alpha^2 r_{11} & 0.1r_{01} + 0.2\alpha r_{11} \\ 0.1r_{01} + 0.2\alpha r_{11} & 0.04r_{11} \end{bmatrix}$$

i.e. $\begin{bmatrix} r_{00} & r_{01} \\ r_{01} & r_{11} \end{bmatrix} = \begin{bmatrix} 0.25r_{00} + \alpha r_{01} + \alpha^2 r_{11} & 0.1r_{01} + 0.2\alpha r_{11} \\ 0.1r_{01} + 0.2\alpha r_{11} & 0.04r_{11} \end{bmatrix} + \sigma^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Equating entries, we obtain

$$r_{00} = 0.25r_{00} + \alpha r_{01} + \alpha^2 r_{11}, \quad r_{01} = 0.1r_{01} + 0.2\alpha r_{11}, \quad r_{11} = 0.04r_{11} + \sigma^2$$

We see $r_{01} = \frac{2}{9}\alpha r_{11}$ and $r_{11} = 1/0.96$

$$3/4\, r_{00} = \frac{2}{9}\alpha^2 r_{11} + \alpha^2 r_{11} = \frac{11}{9}\alpha^2 r_{11}$$

Hence

$$r_{00}/r_{11} = E\{y_k^2\}/E\{v_k^2\} = \frac{4}{3} \times \frac{11}{9}\alpha^2$$

But $r_{00}/r_{11} = 2$, so $\alpha^2 = \cancel{2} \times \frac{3}{\cancel{4}_2} \times \frac{9}{11}$, whence $\alpha = \sqrt{\frac{27}{22}}$

[12]

6 (a) The spectral density of $\{x_k\}$, $\Phi_x(\omega) = \sum_{l=-\infty}^{+\infty} R(l)\, e^{-j\omega l}$,

[2] where $R_x(l) = E\{x_k x_{k-l}\}$ for $l = 0, \pm 1, \dots$

We have $R_y(l) = E\{(a_0 x_k + a_1 x_{k-1})(a_0 x_{k-l} + a_1 x_{k-l-1})\}$

$= (a_0^2 + a_1^2) R_x(l) + a_0 a_1 (R_x(l+1) + R_x(l-1))$.

Hence $\Phi_y(\omega) = \sum_{l=-\infty}^{+\infty} R_y(l)\, e^{-j\omega l}$

$= \sum_{l=-\infty}^{+\infty} [a_0^2 + a_1^2 + a_0 a_1 e^{j\omega} + a_0 a_1 e^{-j\omega}]\, e^{-j\omega l} R_x(l)$

$= (a_0^2 + a_1^2 + a_0 a_1 e^{j\omega} + a_0 a_1 e^{-j\omega}) \sum_{l=-\infty}^{+\infty} R_x(l)\, e^{-j\omega l}$

[6] $= (a_0 + a_1 e^{-j\omega})(a_0 + a_1 e^{+j\omega})\, \Phi_x(\omega) = D(e^{j\omega}) D(e^{-j\omega}) \Phi_x(\omega)$

(b) $$\Phi_y(\omega) = \frac{\left(\frac{17}{16} + \frac{1}{4}[e^{-2j\omega} + e^{+2j\omega}]\right)}{\left(5/4 + \frac{1}{2}[e^{-j\omega} + e^{+j\omega}]\right)\left(\frac{10}{9} + \frac{1}{3}[e^{-j\omega} + e^{+j\omega}]\right)}$$

$$= \left.\frac{\frac{17}{16} + \frac{1}{4}(z^{-2} + z^2)}{\left(5/4 + \frac{1}{2}(z^{-1} + z)\right)\left(\frac{10}{9} + \frac{1}{3}(z^{-1} + z)\right)}\right|_{z = e^{j\omega}}$$

But

$$\frac{17}{16} + \frac{1}{4}(z^{-2} + z^2) = \frac{z^{-2}}{16}[4z^4 + 17z^2 + 4] = \frac{z^{-2}}{16}(4z^2 + 1)(z^2 + 4) = \left(1 + \frac{1}{4}z^{-2}\right)\left(1 + \frac{1}{4}z^2\right)$$

$$\frac{5}{4} + \frac{1}{2}(z^{-1} + z) = \frac{z^{-1}}{4}[2z^2 + 5z + 2] = \frac{z^{-1}}{4}(2z+1)(z+2) = \left(1 + \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z\right)$$

and

$$\frac{10}{9} + \frac{1}{3}(z^{-1} + z) = \frac{z^{-1}}{9}[3z^2 + 10z + 3] = \frac{z^{-1}}{9}(3z+1)(z+3) = \left(1 + \frac{1}{3}z^{-1}\right)\left(1 + \frac{1}{3}z\right)$$

It follows that $\Phi_y(\omega)$ can be factorized

$$\Phi_y(\omega) = \left.\frac{1 + \frac{1}{4}z^{-2}}{(1 + \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})} \cdot \frac{1 + \frac{1}{4}z}{(1 + \frac{1}{2}z)(1 + \frac{1}{3}z)}\right|_{z = e^{j\omega}}$$

Hence the spectral density is 'realised' by the ARMA model

$$y_k = \frac{1 + \frac{1}{4}z^{-2}}{(1 + \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})} e_k,$$ for some zero mean, unit variance, uncorrelated process $\{e_k\}$.

We see that

$y_k = (1 + \frac{1}{4}z^{-2}) x_k$, where $x_k = \frac{1}{(1 + \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})} e_k$

and therefore $a = \frac{1}{4}$ and $l = 2$

Also

[12] $$x_k + \frac{5}{6}x_{k-1} + \frac{1}{6}x_{k-2} = e_k$$