Paper Number(s): **E4.10**

C2.1

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IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2002

MSc and EEE PART IV: M.Eng. and ACGI

PROBABILITY AND STOCHASTIC PROCESSES

Friday, 3 May 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Corrected Copy

Examiners responsible:

First Marker(s): Vinter,R.B.

Second Marker(s): Clark,J.M.C.

Special Instructions for Invigilator: **None**

Information for Students: **None**

1. (a) The ring network of *Figure 1.1*, consisting of eight links, provides two possible paths between terminals A and B. Assume that the links fail independently, each with probability $1-q$, $0 < q < 1$. What is the probability that a packet will be successfully transmitted from A to B?

(Note that terminal A transmits the packet in both directions. B receives the packet if all links transmit in either path.) [10]

(b) A signal $X(\omega)$ comes from one of two sources A or B. (See *Figure 1.2.*) Assume that:

if A is the source, the signal is normally distributed with mean $m_X = -1$ and variance $\sigma^2 = 1$.

if B is the source, the signal is normally distributed with mean $m_X = +1$ and variance $\sigma^2 = 1$.

A signal is received at R only if the switch that links it to its source is closed. One and only one switch is closed at transmission and

10·10

~~P('switch a is closed') = 2 × P('switch b is closed')~~.

P['switch A is closed'] = 2 × P['switch B is closed']

(i) Calculate the probability of the event $\{\omega : X(\omega) \geq -1\}$. [5]

(ii) It is observed that $X(\omega) \geq -1$. What is the most likely source of the signal, A or B? *(The following table includes some relevant values of the distribution function $F(y) = P[Y \leq y]$, for a normally distributed random variable $Y(\omega)$ with zero mean and unit variance.)* [5]

Normal Distribution $N(0,1)$

x	-2	-1	0	+1	+2
$F(x)$	0.02276	0.15866	0.5	0.84134	0.97724

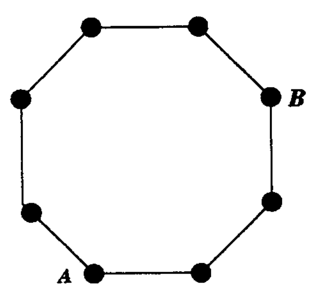


Figure 1.1

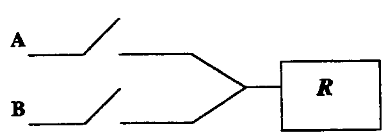


Figure 1.2

2 For a certain communication channel, the received signal $Y(\omega)$ is the transmitted signal $X(\omega)$ corrupted by additive noise

$$Y(\omega) = X(\omega) + N(\omega)\,.$$

Assume that the noise is a zero mean normally distributed random variable with variance σ^2

$$f_N(n) = (2\pi\sigma^2)^{\cancel{-1}\ -1/2} \exp\left(-n^2/2\sigma^2\right)\,.$$ 10·35

Assume also that $X(\omega)$ and $N(\omega)$ are independent.

Determine the conditional probability density of Y given $X(\omega) = x$

$$f_{Y|X}(y|x)\,.$$

[4]

Now suppose $X(\omega)$ is uniformly distributed on $[-\alpha, +\alpha]$ (for some $\alpha > 0$).

Derive a formula for the conditional probability density of X given $Y(\omega) = y$

$$f_{X|Y}(x|y)\,.$$

[10]

Hence derive a formula for the (nonlinear) least squares estimate of X given $Y(\omega) = y$

$$E[X|Y(\omega) = y]\,.$$

Show that, as $\alpha \to \infty$,

$$E[X|Y(\omega) = y] \to y\,.$$

[4]

Comment briefly on this last relationship. [2]

3. (a) The generalized coordinates of a manoeuvring vehicle are represented by the n-vector random variable $X(\omega)$. Motion of the vehicle is affected by the manoeuvre 'mode' $R(\omega)$.

$R(\omega)$ is a discrete random variable, taking values $1, 2, \ldots, n$. Let

$$w_j = P[R = j], \quad j = 1, 2, \ldots, n\,.$$

For $j = 1, 2, \ldots, n$ write

$$F_j(x) = P[X \le x \,|\, R = j]$$

('the conditional probability distribution function of $X(\omega)$ given $R(\omega) = j$'), and denote by m_j and P_j the mean and covariance matrix of $F_j(x)$, respectively.

Derive a formula for the probability distribution of $X(\omega)$, in terms of the $F_j(x)$'s and w_j's. [2]

Show that the mean m and covariance matrix P of $X(\omega)$ are

$$m = \sum_j w_j m_j \quad \text{and} \quad P = \sum_j w_j \left(P_j + (x_j - m)(x_j - m)^T\right)\,.$$

[10]

(b) Henceforth assume that $X(\omega)$ is a scalar random variable and

$$n = 2,\ m_1 = -1,\ m_2 = +1,\ P_1 = P_2 = 0,\ w_1 = w_2 = 1/2\,.$$

Calculate the mean 'range' of the vehicle:

$$E[\,|X|\,]\,. \qquad (3.1)$$

[4]

(c) Sometimes, to simplify calculations, probability distributions are approximated by normal distributions having the same mean and covariance. Examine the effects of this approximation in calculating the mean range. Specifically:

determine the percentage error in the calculation of the mean range, when the normal probability density with (scalar) mean m and variance P,

$$\tilde{f}(x) \;=\; (2\pi P)^{-\frac{1}{2}} \exp\left\{-(x - m)^2/2P\right\},$$

is used in place of F_X to evaluate the expectation in (3.1). [4]

In part (c), you can use the fact that

$$\int_0^\infty (x/\sigma^2)\, exp\left(-x^2/2\sigma^2\right) dx \;=\; 1 \quad for\, \sigma^2 > 0\,.$$

4. (a) A zero-mean scalar random variable $y(\omega)$ is correlated with a zero mean n-vector random variable $\mathbf{x}(\omega)$. Show that the random variable $\hat{\mathbf{x}}(\omega)$ given by

$$\hat{\mathbf{x}}(\omega) = y(\omega)\hat{\mathbf{a}}, \quad \hat{\mathbf{a}} = (E[y^2])^{-1}E[y\mathbf{x}],$$

is the linear least squares estimate of $\mathbf{x}(\omega)$ given $y(\omega)$, in the sense that $\hat{\mathbf{a}}$ minimizes

$$J(\mathbf{a}) := E\left[(\mathbf{x} - y\mathbf{a})^T(\mathbf{x} - y\mathbf{a})\right] .$$

[6]

Derive the following formula for the estimation error covariance matrix

$$\mathrm{cov}\{\mathbf{x} - \hat{\mathbf{x}}\} = E[\mathbf{x}\mathbf{x}^T] - (E[y^2])^{-1}E[y\mathbf{x}]E[y\mathbf{x}^T] .$$

[4]

(b) Consider now the one stage state space system, with scalar output:

$$\begin{aligned} \mathbf{x}_1(\omega) &= A\mathbf{x}_0(\omega) + \mathbf{e}(\omega) \\ y_1(\omega) &= \mathbf{c}^T\mathbf{x}_1(\omega) + v(\omega) . \end{aligned}$$

Here, A is a constant $n \times n$ matrix and $\mathbf{c}$ is a constant n-vector. The n-vector random variables $\mathbf{x}_0$, $\mathbf{e}$ and the scalar random variable v are all uncorrelated.

Furthermore,

$$E[\mathbf{x}_0] = E[\mathbf{e}] = \mathbf{0},\ E[v] = 0,\ E[\mathbf{x}_0\mathbf{x}_0{}^T] = P_0,\ E[\mathbf{e}\mathbf{e}^T] = Q,\ E[v^2] = w .$$

Using part (a), or otherwise, show that the linear least squares estimate of $\mathbf{x}_1$ given y_1 is

$$\hat{\mathbf{x}}_1 = y_1\mathbf{k}$$

where

$$\mathbf{k} = s^{-1}(AP_0A^T + Q)\mathbf{c} \quad \text{and} \quad s = \left(\mathbf{c}^T(AP_0A^T + Q)\mathbf{c} + w\right) .$$

[6]

Show, furthermore, that the covariance matrix of $\mathbf{x}(\omega) - \hat{\mathbf{x}}(\omega)$ is

$$P_1 = (AP_0A^T + Q)\left(I - s^{-1}\mathbf{c}\mathbf{c}^T(AP_0A^T + Q)\right) .$$

[4]

5. (a) Consider the scalar Auto-Regressive Moving Average (ARMA) process $\{y_k\}$, generated by the difference equation

$$y_k + g y_{k-2} \;=\; e_k + h e_{k-1}\,,$$

in which $\{e_k\}$ is a sequence of uncorrelated, zero mean random variables with variance σ^2. g, $|g| < 1$, and h are constants.

Show that the covariance function $R_y(k)$, for $k = 0$, is

$$R_y(0) \;=\; \frac{1+h^2}{1-g^2}\sigma^2\,.$$

Determine also $R_y(1)$ and $R_y(2)$. [12]

(b) Now consider the controlled Auto-Regressive process $\{y_k\}$

$$y_k - a y_{k-1} = e_k + u_{k-2}, \tag{5.1}$$

in which e_k is as before and a is a constant, with $|a| < 1$. The control u_k, which depends on present and past values of $\{y_k\}$, is chosen to improve the statistical properties of the process $\{y_k\}$. Notice that there is a two sample period delay in control implementation.

For this system, a 'minimum variance' controller has the structure:

$$u_k + a u_{k-1} \;=\; K y_k, \tag{5.2}$$

in which K is a design parameter.

Derive the ARMA model for the process $\{y_k\}$ which results when the minimum variance controller (5.2) is inserted into ($\cancel{p}$5.1). What conditions must K satisfy for this ARMA model to be stable? 10 / 10

Determine the value of K, satisfying the stability condition, which minimizes the output covariance:

$$E[y_k^2]\,.$$

Show that, for this choice of K, $\{y_k\}$ is a Moving Average process. [8]

6. Define the spectral density $\Phi(\omega)$ of a stationary, second order, zero mean, scalar stochastic process $\{y_k\}$. What conditions must $\Phi(\omega)$ satisfy if $\{y_k\}$ is to be the output of an Auto-Regressive Moving Average model

$$A(z^{-1})y_k = B(z^{-1})e_k \text{ ?} \tag{6.1}$$

[4]

Here A and B are polynomials in the delay operator z^{-1} and $\{e_k\}$ is a sequence of zero mean, unit variance, uncorrelated scalar random ~~variables,~~ variables. FULL STOP 10·45

Consider now the covariance function

$$R(k) = c_1 e^{-\lambda_1|k|} + c_2 e^{-\lambda_2|k|} \quad k = \ldots, -1, 0, +1, \ldots,$$

in which c_1, c_2, λ_1 and λ_2 are positive constants. Show that the corresponding spectral density function $\Phi(\omega)$ is

$$\Phi(\omega) = \sum_{i=1}^{2} \frac{c_i(1 - e^{-2\lambda_i})}{(1 - e^{-\lambda_i}e^{-j\omega})(1 - e^{-\lambda_i}e^{+j\omega})}\ .$$

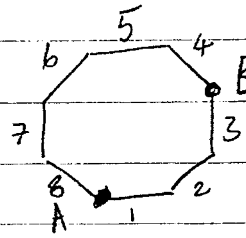
[8]

Now set

$$c_1 = 4/3,\ \lambda_1 = \log_e(2),\ c_2 = 9/8, \text{ and } \lambda_2 = \log_e(3),$$

For these values of the constants, determine an ARMA model (6.1) whose output $\{y_k\}$ has the covariance function $R(k)$. [8]

Note: in this question ω denotes a frequency, not a point in the sample space.



1(a) Let S_i denote "link i transmits".
The, if P_{tr} is the probability of transmission

$$P_{tr} = P\left((S_1 \cap S_2 \cap S_3) \cup (S_4 \cap .. \cap S_5) \right)$$

Hence $1 - P_{tr} = P\left(\overline{S_1 \cap S_2 \cap S_3} \cap \overline{S_4 \cap .. \cap S_5} \right)$

Independence of the S_i's implies $\overline{S_1 \cap S_2 \cap S_3}$ and $\overline{S_4 \cap .. \cap S_5}$ are independent, so

$$1 - P_{tr} = P(\overline{S_1 \cap S_2 \cap S_3}) \cdot P(\overline{S_4 \cap .. \cap S_8}) = (1 - P(S_1 \cap .. \cap S_3))(1 - P(S_4 \cap .. \cap S_8))$$
$$= (1 - P(S_1) \times .. \times P(S_3))(1 - P(S_4) \times .. \times P(S_8)) = (1 - q^3)(1 - q^5)$$

Hence,

10 probability of transmission $= 1 - (1 - q^3)(1 - q^5)$

(b) Let A denote "A is source", etc. Then

$$P[X(\omega) \geq -1] = P[X(\omega) \geq -1 \mid A] . P[A] + P[X \geq -1] P[B]$$

Since $P[A] + P[B] = 1$ and $P[A] = 2 * P[B]$, $P[A] = 2/3$, $P[B] = 1/3$

Since $f_{X|A} \sim N(-1, 1)$, $P[X(\omega) \geq -1 \mid A] = 0.5$

Since $f_{X|B} \sim N(+1, 1)$

$$P[X(\omega) \geq -1 \mid B] = 1 - \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{-1} e^{-\frac{1}{2}|t-1|^2} dt = 1 - \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{-2} e^{-t^2/2} dt$$
$$= 1 - 0.02276 = 0.97724$$

5 Hence $P[X(\omega) \geq -1] = 0.5 \times \frac{2}{3} + 0.97724 \times \frac{1}{3} = 0.65908$

By Bayes' Rule:

$$P[A \mid X(\omega) \geq -1] = \frac{P[X(\omega) \geq -1 \mid A] \, P[A]}{P[X(\omega) \geq -1]} = \frac{0.5 . \frac{2}{3}}{0.65908} = 0.50576$$

and

$$P[B \mid X(\omega) \geq -1] = \frac{P[X(\omega) \geq -1 \mid B] \, P[B]}{P[X(\omega) \geq -1]} = \frac{0.97724 . \frac{1}{3}}{0.65908} = 0.49424$$

We see that

$$P[A \mid X(\omega) \geq -1] > P[B \mid X(\omega) \geq -1].$$

5 It is therefore more likely that the source was "A"

2. $F_{Y/X}(y|x) = P[Y \le y \mid 'X(\omega)=x'] = P[X+N \le y \mid 'X(\omega)=x']$
$= P[N \le y-x \mid 'X(\omega)=x'] = P[N \le y-x]$
(by independence)
$= F_N(y-x)$

4 It follows $\underbrace{f_{Y/X}(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(y-x)^2}{\sigma^2}\right)}$

We know

$$f_{X/Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y} = \frac{f_{Y/X}(y|x) \times f_X(x)}{f_Y(y)}$$

But $f_X(x) = \begin{cases} 1/2\alpha & -\alpha \le x \le +\alpha \\ 0 & \text{otherwise} \end{cases}$

Also, for each y

$$\int_{-\infty}^{+\infty} f_{X/Y}(x|y)\,dx = 1$$

This implies

$$\frac{1}{2\alpha}\int_{-\alpha}^{+\alpha} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(y-x')^2}{\sigma^2}\right)dx' = f_Y(y)$$

It follows

10 $$f_{X/Y}(x|y) = \begin{cases} \dfrac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}|y-x|^2\right)}{\int_{-\alpha}^{+\alpha} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}|y-x'|^2\right)dx'} & \text{if } -\alpha \le x \le +\alpha \\ 0 & \text{otherwise} \end{cases}$$

The conditional expectation of $X(\omega)$ given $Y(\omega)=y$ is

4 $$\underbrace{E[X(\omega) \mid Y(\omega)=y] = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\alpha}^{+\alpha} x \exp\left(-\frac{1}{2\sigma^2}|y-x|^2\right)dx}{\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\alpha}^{+\alpha} \exp\left(-\frac{1}{2\sigma^2}|y-x'|^2\right)dx'}} = \frac{a}{b}$$

2 Notice that $a \to \frac{1}{2\pi\sigma^2}\int_{-\infty}^{+\infty} x \exp\left(-\frac{1}{2\sigma^2}|y-x|^2\right)dx = y$
$b \to \frac{1}{2\pi\sigma^2}\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}|y-x|^2\right)dx = 1$

as $\alpha \to \infty$ (by properties of the normal density). So $\underbrace{E[X/Y] \to y}$

Note: '$\alpha \to \infty$' implies we have no prior information about $X(\omega)$. It is natural, in these circumstances to estimate $X(\omega)$ as the value

3 (a) We have

$$F_X(x) = P[X \le x] = \sum_j P[X \le x \text{ and } R=j]$$

$$= \sum_j P[X \le x \mid R=j]\,P[R=j] = \underbrace{\sum_j F_j(x)\,\omega_j}.$$

2 So $m = \int x\, dF_X(x) = \sum_j \omega_j \int x\, dF_j(x) = \underbrace{\sum_j \omega_j m_j}$

Also,

$$P = \int (x-m)(x-m)^T dF_X(x)$$

$$= \sum_j \omega_j \int (x-m)(x-m)^T dF_j(x)$$

$$= \sum_j \omega_j \Big[\int x x^T dF_j(x) - \int x\, dF_j(x) . m^T - m \int x^T dF_j(x) + m m^T \int dF_j(x) \Big]$$

$$= \sum_j \omega_j \left(P_j + m_j m_j^T - m m_j^T - m m_j^T + m m^T \right)$$

10 $$= \underbrace{\sum_j \omega_j \left(P_j + (m_j - m)(m_j - m)^T \right)}$$

(b) When $m_1 = -1,\ m_2 = +1,\ P_1 = P_2 = 0,\ \omega_1 = \omega_2 = \tfrac{1}{2}$

$m = \frac{1}{2}(-1) + \frac{1}{2}(+1) = 0$ and

$$P = \tfrac{1}{2}(1-0)^2 + \tfrac{1}{2}(1-0)^2 = 1$$

Also

$X(\omega)$ is a discrete RV : $P(X=-1) = P(X=+1) = \tfrac{1}{2}$.

It follows

4 mean range $= E|X| = |-1| \cdot \frac{1}{2} + |+1| \cdot \frac{1}{2} = \underbrace{1}$

If we use the normal density to evaluate 'mean range'

$$\text{approx. mean range} = \int_{-\infty}^{+\infty} |x| . \frac{1}{(2\pi)^{\frac{1}{2}}} \exp\left(-\tfrac{1}{2} x^2\right) dx$$

$$= \frac{2}{(2\pi)^{\frac{1}{2}}} \int_0^{+\infty} |x| \exp\left(-\tfrac{1}{2}x^2\right) dx = \sqrt{\frac{2}{\pi}}$$

4 $$\% \text{ error} = \frac{1 - \sqrt{\frac{2}{\pi}}}{1} \times 100 = \underbrace{20.2115}\ \%$$

4 (a) $J(a) = E[(\underline{x} - y\underline{a})^T(\underline{x} - y\underline{a})] = E\{x^T x\} - 2\sum a_i E[y x_i]$
$+ (\sum_i a_i) E[y^2]$

Since $J(\hat{a}_1, \ldots, \hat{a}_{i-1}, a_i, \hat{a}_{i+1}, \ldots, \hat{a}_n)$ is minimized at $a_i = \hat{a}_i$, for each i

$0 = \partial/\partial a_i \, J(\hat{a}) = 2E[y x_i] + 2\hat{a}_i E[y^2]$. Hence

$$\hat{a}_i = E[y^2]^{-1} E[y x_i], \quad i = 1, \ldots, n.$$

These relationships can be expressed:

6 $$\hat{a} = E[y^2]^{-1} E[y\underline{x}]$$

Since $x, \hat{x}$ and y have zero mean,

$$\mathrm{cov}\{x - \hat{x}\} = E[(x-\hat{x})(x-\hat{x})^T]$$
$$= E[(x - y(E[y^2])^{-1}E[y\underline{x}])(x^T - y(E[y^2])^{-1}E[y\underline{x}^T])]$$
$$= E[xx^T] - 2(E[y^2])^{-1}E[y\underline{x}]E[y\underline{x}^T] + (E[y^2])^{-1}E[y\underline{x}]E[y\underline{x}^T]$$

4 $$= \mathrm{cov}\{x\} - (E[y^2])^{-1}E[y\underline{x}]E[y\underline{x}^T]$$

(b) System equations: $\underline{x}_1 = A\underline{x}_0 + e$ and $y_1 = c^T\underline{x}_1 + v$

We must evaluate $E\{y_1^2\}$ and $E\{y_1\underline{x}_1\}$. Since x_1 and v are uncorrelated and x_0 and e are uncorrelated

$$E\{y_1^2\} = c^T E\{x_1 x_1^T\} c + 0 + E\{v^2\}.$$

But $E\{x_1 x_1^T\} = A E\{\underline{x}_0 x_0^T\} A^T + E[ee^T] = AP_0A^T + Q$

Hence $E\{y_1^2\} = c^T(AP_0A^T + Q)c + W$.

Also

$$E\{y_1\underline{x}_1\} = E\{x_1x_1^T\}c + 0 = (AP_0A^T + Q)c$$

Furthermore,

$$\mathrm{cov}\{x_1 - \hat{x}_1\} = E\{x_1x_1^T\} - (E\{y^2\})^{-1}E\{y_1x_1\} \cdot E\{y_1x_1^T\}$$
$$= AP_0A^T + Q - [c^T(AP_0A^T+Q)c + w](AP_0A^T+Q)cc^T(AP_0A^T+Q).$$

By part (a), $\hat{x} = y_1 k$, where

$$k = (E[y_1^2])^{-1}E[y_1\underline{x}_1] = s^{-1}(AP_0A^T + Q)c$$

6 and $s = c^T(AP_0A^T + Q)c + w$

Also

$$\mathrm{cov}(x_1 - \hat{x}_1) = E[x_1x_1^T] - (E[y_1^2])^{-1}E\{y_1\underline{x}_1\}E[y_1\underline{x}_1^T]$$

4 $$= (AP_0A^T + Q)[I - s^{-1}cc^T(AP_0A^T + Q)]$$

5(a) $y_k + g\, y_{k-2} = e_k + h e_{k-1}$

$E\{\ldots \times y_k\} \Rightarrow R_y(0) + g R_y(2) = R_{ye}(0) + h R_{ye}(1)$

$E\{\ldots \times y_{k-1}\} \Rightarrow R_y(1) + g R_y(1) = 0 + h R_{ye}(0)$

$E\{\ldots \times y_{k-2}\} \Rightarrow R_y(2) + g R_y(0) = 0 + 0.$

(We have used the facts that $R_y(1) = R_y(-1)$ and y_k is uncorrelated with e_j, $j > k$.)

$E\{\ldots \times e_k\} \Rightarrow R_{ye}(0) = \sigma^2 + 0$

$E\{\ldots \times e_{k-1}\} \Rightarrow R_{ye}(1) + 0 = h\sigma^2$

Also, $E\{\ldots \times y_{k-j}\}$ $(j > 2) \Rightarrow R_y(j) + g R_y(j-2).$ — (*)

From these relationships

$(1+g)\, R(1) = h\sigma^2 \Rightarrow R(1) = \frac{h}{1+g}\sigma^2$

$\left.\begin{array}{l} R_y(0) + g R_y(2) = (1+h^2)\sigma^2 \\ R_y(2) + g R_y(0) = 0 \end{array}\right\} \Rightarrow R_y(0)(1-g^2) = (1+h^2)\sigma^2$

12 Hence $R_y(0) = \frac{1+h^2}{1-g^2}\sigma^2$ and $R_y(2) = \frac{-g(1+h^2)}{(1-g^2)}\sigma^2$

From (*) $R_y(k) = \begin{cases} -g^{\left|\frac{k}{2}\right|} \cdot \frac{1+h^2}{(1-g^2)} \cdot \sigma^2 & \text{for } k \text{ even} \\ -g^{\frac{|k|+1}{2}} \cdot \frac{h}{1+g} \cdot \sigma^2 & \text{for } k \text{ odd} \end{cases}$

(b) Inserting the control $u = \frac{-K}{(1+az^{-1})} y$ into the system equations gives

$(1-az^{-1})\, y_k = e_k - \frac{K}{(1+az^{-1})} y_k.$

Rationalizing: $(1-az^{-1})(1+az^{-1})\, y_k = (1+az^{-1})\, e_k - K y_k.$

Hence $y_k - (a-K)\, y_{k-2} = e_k + a e_{k-1}$ — (**)

2 This is stable if $|K-a| < 1$

By (a),

$R_y(0) = \left.\frac{1+h^2}{1-g^2}\right|_{h=a,\, g=K-a} = \frac{1+a^2}{1-(K-a)^2}$

4 Since $|K-a| < 1$, the minimizing K is $K = a$

For this choice of a, from (**),

$y_k = e_k + a e_{k-1}.$

2 According to this relationship, $\{y_k\}$ is a moving average process.

6 If $R(\ell)$ is the covariance function of $\{y_k\}$, i.e. $R(\ell) = E\{y_k y_{k-\ell}\}$ then the spectral density is

2 $$\Phi(\omega) = \sum_{\ell=-\infty}^{+\infty} R(\ell)\, e^{-j\omega \ell}$$

$\Phi(\omega)$ is the spectral density of an ARMA process if and only if it can be factorized

2 $$\Phi(\omega) = D(z)\,D(z^{-1})\Big|_{z=e^{-j\omega}}$$

in which $D(z)$ is a rational function of z.

If $R(k) = c_1 e^{-\lambda_1 |k|} + c_2 e^{-\lambda_2 |k|}$. Then

$$\Phi(\omega) = c_1 \sum_{k=-\infty}^{+\infty} e^{-\lambda_1 |k|} \cdot e^{-j\omega k} + \ldots \text{ (same but with } c_2, \lambda_2)$$

$$= c_1 \left(\sum_{k=0}^{\infty} e^{-(\lambda_1 + j\omega)k} + \sum_{k=1}^{\infty} e^{-(\lambda_1 - j\omega)k} \right) + \ldots$$

$$= c_1 \left(\frac{1}{1 - e^{-\lambda_1} \cdot e^{-j\omega}} + \frac{1}{1 - e^{-\lambda_1} \cdot e^{+j\omega}} - 1 \right) + \ldots$$

8 $$= \underbrace{\frac{c_1 (1 - e^{-2\lambda_1})}{(1 - e^{-\lambda_1} \cdot e^{-j\omega})(1 - e^{-\lambda_1} e^{+j\omega})} + \frac{c_2 (1 - e^{-2\lambda_2})}{(1 - e^{-\lambda_2} \cdot e^{-j\omega})(1 - e^{-\lambda_2} e^{+j\omega})}}$$

when $c_1 = \frac{4}{3}$, $\lambda_1 = \ln(2)$, $c_2 = \frac{9}{8}$, $\lambda_2 = \ln(3)$

$$\Phi(\omega) = \frac{1}{(1 - \frac{1}{2} e^{-j\omega})(1 - \frac{1}{2} e^{+j\omega})} + \frac{1}{(1 - \frac{1}{3} e^{-j\omega})(1 - \frac{1}{3} e^{+j\omega})}$$

Hence $\Phi(\omega) = \Psi(z)\,\Psi(z^{-1}) \mid z = e^{-j\omega}$, with

$$\Psi(z) = \frac{(1 - \frac{1}{3} z)(1 - \frac{1}{3} z^{-1}) + (1 - \frac{1}{2} z)(1 - \frac{1}{2} z^{-1})}{(1 - \frac{1}{2} z^{-1})(1 - \frac{1}{3} z^{-1})(1 - \frac{1}{2} z)(1 - \frac{1}{3} z)}$$

$$= \frac{85 - 30 z^{-1} - 30 z}{(2 - z^{-1})(3 - z^{-1})(2 - z)(3 - z)}$$

The roots of $30 z^2 - 85 z + 30$ are $z = 2.419972$ and $\frac{1}{2.419972}$

so $$\Psi(z) = (3.520915)^2 \frac{(2.419972 - z^{-1})}{(2 - z^{-1})(3 - z^{-1})} \cdot \frac{(2.419972 - z)}{(2 - z)(3 - z^{-1})}$$

It follows that the covariance function is realized by the ARMA model

8 $$\underbrace{(2 - z^{-1})(3 - z^{-1}) y_k = 3.520915 \, (2.419972 - z^{-1})\, e_k}$$