Master – 15/4/03

<u>Optoelectronics 2003 - Solutions</u>

1a) The eigenvalue equation for TE modes in an asymmetric guide is:

$$\tan(\kappa d) = \kappa\{\gamma + \delta\} / \{\kappa^2 - \gamma\delta\}$$

Here $\kappa = \sqrt{\{n_1^2 k_0^2 - \beta^2\}}$, $\gamma = \sqrt{\{\beta^2 - n_2^2 k_0^2\}}$ and $\delta = \sqrt{\{\beta^2 - n_3^2 k_0^2\}}$, β is the propagation constant and $k_0 = 2\pi/\lambda$, where λ is the wavelength.

When $n_2 = n_3$, $\gamma = \delta$, so the eigenvalue equation reduces to:

$$\tan(\kappa d) = 2\kappa\gamma / \{\kappa^2 - \gamma^2\}$$

Dividing top and bottom by 1) κ^2 and 2) γ^2, we may obtain alternatively:

$$1)\ \tan(\kappa d) = 2(\gamma/\kappa) / \{1 - (\gamma/\kappa)^2\} \quad \text{and} \quad 2)\ \tan(\kappa d) = -2(\kappa/\gamma) / \{1 - (\kappa/\gamma)^2\}$$

From the double angle formulae for the tan function, we know that:

$$\tan(2A) = 2\tan(A) / \{1 - \tan^2(A)\}$$

Comparing with Equations 1) and 2) above we can identify κh with 2A to obtain:

$$3)\ \tan(\kappa d/2) = (\gamma/\kappa) \quad \text{and} \quad 4)\ \tan(\kappa d/2) = -(\kappa/\gamma)$$

[4]

b) At cutoff, the ray angle inside the guide lies exactly at the critical angle, so that total internal reflection no longer occurs. The decay constant γ becomes zero, so that the evanescent field no longer falls away rapidly near the guide.

[2]

If $\gamma = 0$ at cutoff, then $\beta = n_2 k_0$ and $\kappa = k_0\sqrt{\{n_1^2 - n_2^2\}}$

[3]

From equations 3) and 4) we then obtain:

$$5)\ \tan(\kappa d/2) = 0 \quad \text{and} \quad 6)\ \tan(\kappa d/2) = -\infty$$

The separate cutoff conditions are therefore:

$$7)\ \kappa d/2 = 0, \pi, 2\pi, \ldots \quad \text{and} \quad 8)\ \kappa d/2 = \pi/2, 3\pi/2, 5\pi/2 \ldots$$

Equations 7) and 8) may be grouped together to obtain a general condition in the form:

$$9)\ \kappa d/2 = \nu\pi/2, \text{ where } \nu = 0, 1, 2, \ldots$$

Using the value for κ found above, we then obtain:

$$10)\ k_0 d/2 \sqrt{\{n_1^2 - n_2^2\}} = \nu\pi/2 \quad \text{or} \quad 11)\ (2d/\lambda)\sqrt{\{n_1^2 - n_2^2\}} = \nu$$

as the general cutoff condition.

[5]

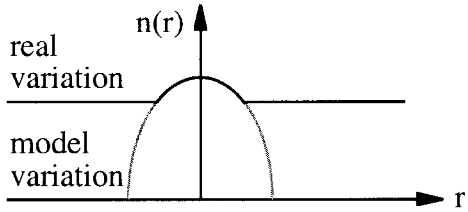
c) For the second order mode cutoff, we require $\nu = 1$ (the lowest order mode is $\nu = 0$). Equation 11 may then be used to complete Table I as shown below:

Sample	n_1	n_2	d (μm)	λ_{cutoff} (μm)*
#1	1.500	**1.499**	5.8	0.6330
#2	1.510	1.500	**1.8(24)**	0.6330
#3	1.505	1.500	3.8	**0.93(15)**

*Cutoff wavelength for the second lowest-order mode.

[6]

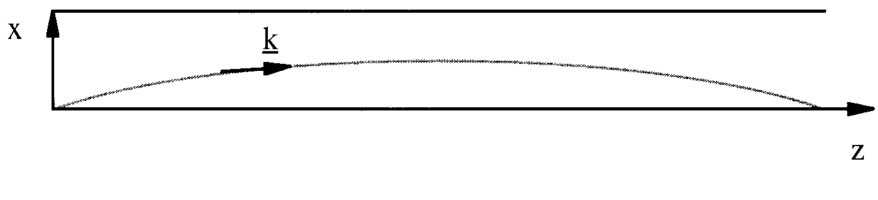
2a) The real and model index variations are as shown below.



[4]

b) Assuming that $r = x$ and $r_0 = x_0$, we first put $n(x) = n_0 \sqrt{\{1 - (x/x_0)^2\}}$.

Now, the direction of the ray at any point (x, z) may be then defined by a vector $\underline{k}$, such that $|\underline{k}|$ is equal to the local value of the propagation constant, $k_0 n(x)$, where $k_0 = 2\pi/\lambda$. Similarly, the local slope of the trajectory may be found as $dx/dz = k_x/k_z$, where k_x and k_z are the x- and z-components of $\underline{k}$.



However, if the entire trajectory corresponds to a guided mode, we may also set $k_z = \beta$, where β is the modal propagation constant. We then obtain $k_x = \sqrt{(k_0^2 n^2 - \beta^2)}$.

Substituting, we then get $\quad k_x = \sqrt{\{(k_0^2 n_0^2 - \beta^2) - k_0^2 n_0^2 (x/x_0)^2\}}$

This may be re-arranged as $\quad \beta^2 (dx/dz)^2 = (k_0^2 n_0^2 - \beta^2) - k_0^2 n_0^2 (x/x_0)^2$

We now guess the solution $x = A \sin(Bz + C)$. Differentiating and substituting gives:

$$(k_0^2 n_0^2 - \beta^2) - (k_0^2 n_0^2 A^2/x_0^2)\sin^2(Bz + C) - \beta^2 A^2 B^2 \cos^2(Bz + C) = 0$$

The only way to satisfy the equation above for all z is by removing the $\sin^2$ and $\cos^2$ terms. This can be done if their coefficients are equal, since $\sin^2\theta + \cos^2\theta = 1$.

This requires $k_0^2 n_0^2 A^2/x_0^2 = \beta^2 A^2 B^2$ or $\quad B = k_0 n_0/\beta x_0$

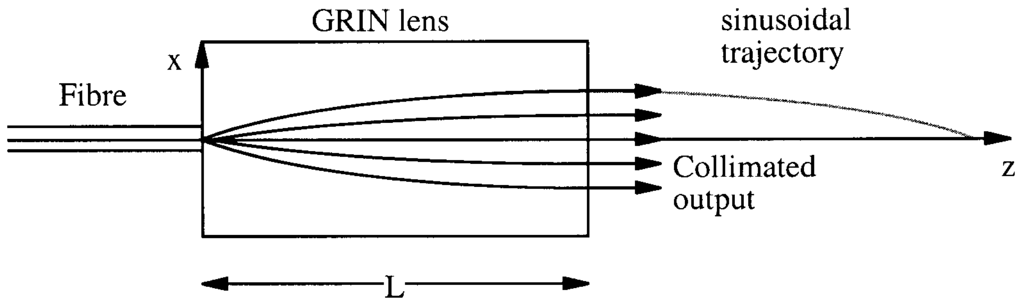
And $(k_0^2 n_0^2 - \beta^2) - (k_0^2 n_0^2 A^2/x_0^2) = 0$ so that $\quad A = x_0 \sqrt{(1 - \beta^2/k_0^2 n_0^2)}$

Because values can be found for A and B, the assumed solution is a valid one.

[10]

c) For small index differences forming the guide, $\beta \approx k_0 n_0$ and $B \approx 1/x_0$, so $x \approx A \sin(z/x_0 + C)$. All rays then have approximately the same periodicity, and differ only in their amplitude A and phase offset C.

A quarter pitch gradient index rod lens has its length L chosen so that $L/x_0 = \pi/2$. Each ray trajectory through the lens then has the form of a quarter period of a sinusoid. An axially symmetric input (obtained from an optical fibre, for example) will then give rise to a set of rays that travel in the z-direction as similar sinusoids and then emerge parallel, i.e. as a collimated beam. Alternatively, a parallel beam may be brought to a point focus.



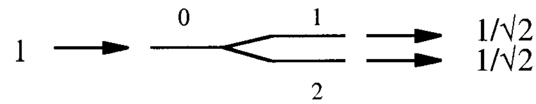
[6]

3a) Assuming that $A_0 = 1$, $P_0 = 1$. The power will be divided equally between the two outputs, so that $P_1 = 1/2$ and $P_2 = 1/2$.

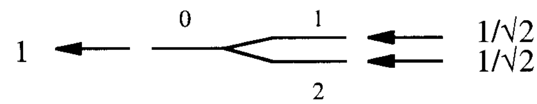
[2]

The amplitudes are therefore $A_1 = 1/\sqrt{2}$ and $A_2 = 1/\sqrt{2}$.

[2]

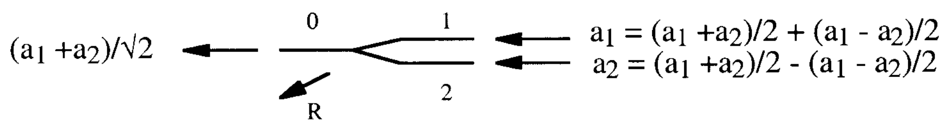


b) Reversing the above result, we can see that a symmetric input of $A_1 = 1/\sqrt{2}$ and $A_2 = 1/\sqrt{2}$ will give an output of $A_0 = 1$. A symmetric input of of $A_1 = a$ and $A_2 = a$ will therefore give an output of $A_0 = a\sqrt{2}$.



An input of $A_1 = a_1$ and $A_2 = a_2$ may be decomposed into symmetric transmitted and antisymmetric radiated components as shown below. The symmetric component has amplitude $a = (a_1 + a_2)/2$. The output amplitude is then $A_0 = a\sqrt{2} = (a_1 + a_2)/\sqrt{2}$.

[2]



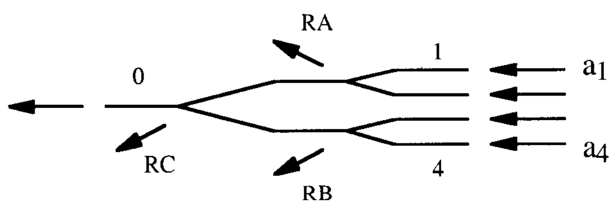
The output power is $P_0 = (a_1 + a_2)^2/2$.

[2]

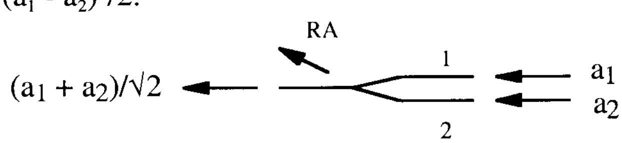
The input power is $P = a_1^2 + a_2^2$. Since power must be conserved, the radiated power must be $P_R = P - P_0 = (a_1 - a_2)^2/2$.

[2]

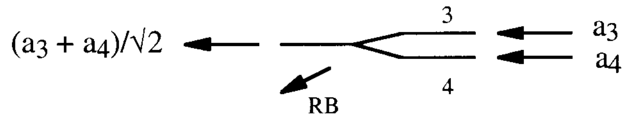
c) The two-stage tree may be analysed by considering each Y-junction separately.



Considering the upper right-hand Y-junction first, we can see from the results above that the guided output amplitude must be $(a_1 + a_2)/\sqrt{2}$. The radiated power must be $P_{RA} = (a_1 - a_2)^2/2$.



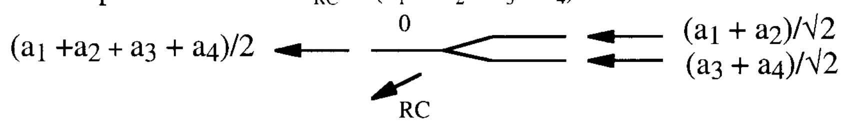
Considering now the lower right-hand Y-junction, the guided output amplitude must be $(a_3 + a_4)/\sqrt{2}$. The radiated power must be $P_{RB} = (a_3 - a_4)^2/2$.



Finally, for the left-hand Y-junction, the inputs must be $(a_1 + a_2)/\sqrt{2}$ and $(a_3 + a_4)/\sqrt{2}$. The guided output amplitude must therefore be $A_0 = (a_1 + a_2 + a_3 + a_4)/2$.

[2]

The radiated power must be $P_{RC} = (a_1 + a_2 - a_3 - a_4)^2/4$.



For a 1-stage tree, we have $M = 1$, $N = 2$, $A_0 = (1/\sqrt{2}) \Sigma a_i$

For a 2-stage tree, we have $M = 2$, $N = 4$, $A_0 = (1/2) \Sigma a_i$

For an M stage tree, we would therefore guess $A_0 = (1/\sqrt{N}) \Sigma a_i$.

[2]

The input power must be $P = a_1^2 + a_2^2 + a_3^2 + a_4^2$.

The guided output power is:

$P_0 = (a_1 + a_2 + a_3 + a_4)^2/4$, or:

$P_0 = (a_1^2 + a_2^2 + a_3^2 + a_4^2 + 2a_1a_2 + 2a_1a_3 + 2a_1a_4 + 2a_2a_3 + 2a_2a_4 + 2a_3a_4)/4$.

The radiated output powers are

$P_{RA} = (a_1 - a_2)^2/2$, or:

$P_{RA} = (2a_1^2 + 2a_2^2 - 4a_1a_2)/4$

$P_{RB} = (a_3 - a_4)^2/2$, or:

$P_{RB} = (2a_3^2 + 2a_4^2 - 4a_3a_4)/4$.

[2]

$P_{RC} = (a_1 + a_2 - a_3 - a_4)^2/4$, or:

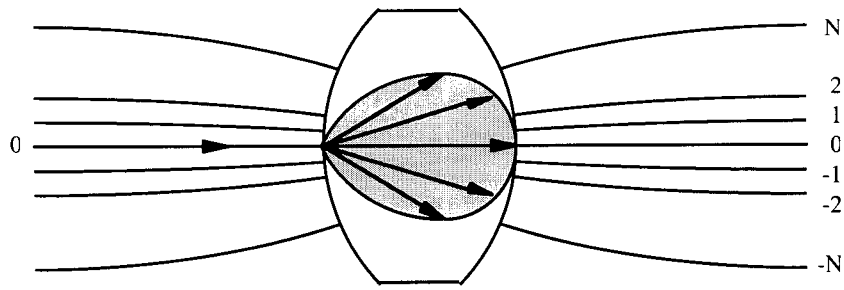
$P_{RC} = (a_1^2 + a_2^2 + a_3^2 + a_4^2 + 2a_1a_2 - 2a_1a_3 - 2a_1a_4 - 2a_2a_3 - 2a_2a_4 + 2a_3a_4)/4$.

[2]

Summing these powers, we obtain $P_0 + P_{RA} + P_{RB} + P_{RC} = P$ as required.

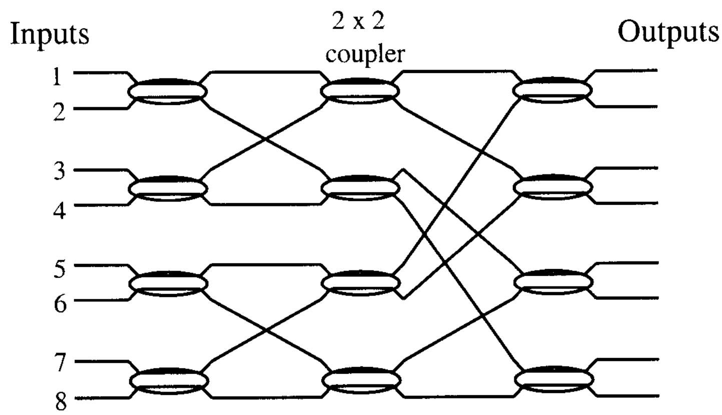
[2]

4a) The *broadcast* function is obtained in a star coupler as shown below. When light from (say) the 0^{th} input reaches the planar guide region, it is no longer confined in the lateral direction and diverges as it travels towards the output guides. Because the polar radiation pattern is slowly varying, this light illuminates the output guides roughly equally, so that the input power is divided approximately equally amongst the outputs. A similar situation arises for the other possible inputs. Consequently, an input to any guide is broadcast equally to all output guides.



[3]

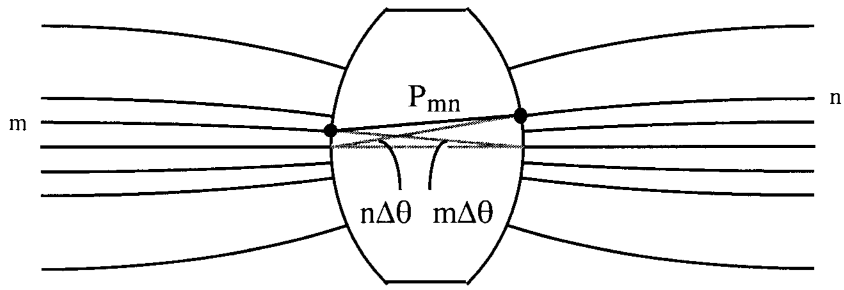
A directional coupler based star achieves the same result using a ladder network of 2 x 2 splitters. An N-stage ladder can be used to generate a 2^N x 2^N star, and the figure below shows a three-stage ladder arranged as an 8 x 8 star.



Somewhat surprisingly, the radiative star has far better performance than the coupler-based star, because variations in the accuracy of the power division of each coupler rapidly degrade the uniformity of the overall power distribution. These variations can arise from manufacturing, or changes in wavelength or polarisation. The radiative star is also simpler to lay out in mask-making.

[5]

b) For the path P_{mn} between the m^{th} input and the n^{th} output as shown below, the lengths measured in the x- and y-directions are:



$L_x = R - R\{1 - (1 - \cos(n\Delta\theta)\} - R\{1 - (1 - \cos(m\Delta\theta)\}$ and $L_y = R\sin(m\Delta\theta) - R\sin(n\Delta\theta)$

Assuming small angles, these expressions can be approximated as:

$L_x \approx R - R\, n^2\Delta\theta^2/2 - R\, m^2\Delta\theta^2/2$ and $L_y \approx R\,(n\Delta\theta) - R\,(m\Delta\theta)$

So the distance P_{ij} is: $P_{mn}{}^2 = L_x{}^2 + L_y{}^2$
Or: $P_{mn}{}^2 \approx R^2\{1 - n^2\Delta\theta^2/2 - m^2\Delta\theta^2/2\}^2 + R^2\{n\Delta\theta - m\Delta\theta\}^2$
Or: $P_{mn}{}^2 \approx R^2\{1 - n^2\Delta\theta^2 - m^2\Delta\theta^2\} + R^2\{n^2\Delta\theta^2 - 2mn\Delta\theta^2 + m^2\Delta\theta^2\}$
Or: $P_{mn}{}^2 \approx R^2\{1 - 2mn\Delta\theta^2\}$
So that: $P_{mn} \approx R\{1 - mn\Delta\theta^2\}$

[7]

Assuming that the propagation constant is β, the relationship between the input amplitude on guide m and the output amplitude on guide n is:

$A_n = A_m \exp\{-j\beta P_{mn}\} = A_m \exp\{-j\beta R\} \exp\{+jmn\beta R\Delta\theta^2\}$

If all inputs are excited simultaneously, we can obtain the output amplitude from guide n by assuming an equal power division during radiation, and summing the components reaching output n from each of the N inputs as:

$A_n = C \exp\{-j\phi\} \sum_{m=-N}^{m=+N} A_m \exp\{+jmn\alpha\}$
where $C = 1/\sqrt{\{2N + 1\}}$, $\phi = \beta R$ and $\alpha = \beta R\Delta\theta^2$

Apart from the unimportant (constant) phase term $\exp\{-j\phi\}$, this expression has the form of a discrete Fourier transform.

[5]

5a) The double heterostructure are the following three-layers:
p-$Ga_{0.7}Al_{0.3}As$ (3) - GaAs (4) - n-$Ga_{0.7}Al_{0.3}As$ (5).

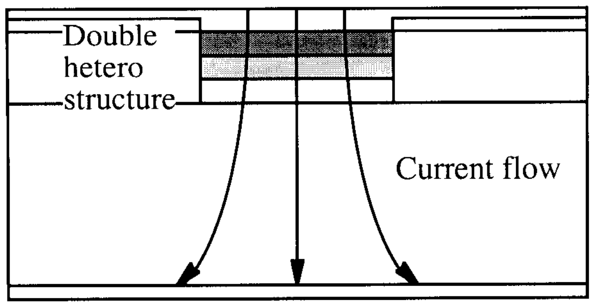
[3]

The function of layer 6 is two-fold: it has low refractive index, and so provides lateral confinement for the channel guide, and it has high-resistivity, and so reduces spreading of the injection current before it has passed through the heterostructure.

[2]

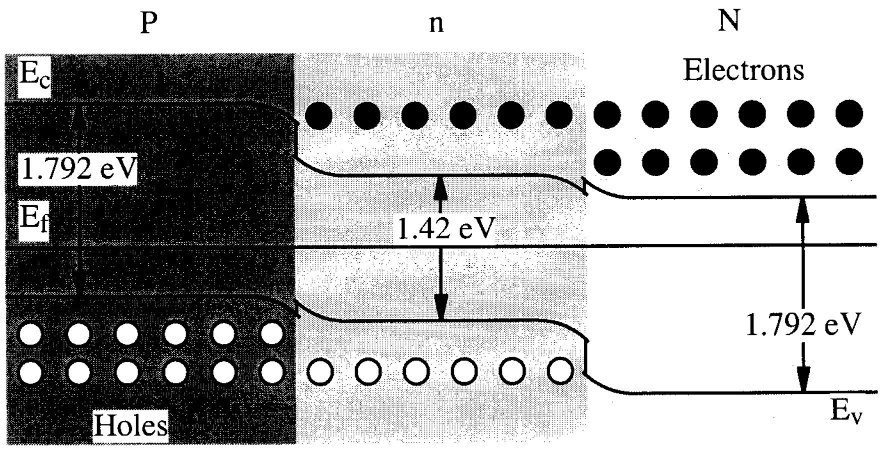
The function of layer 7 is to provide an ohmic contact to the Al metal.

[1]



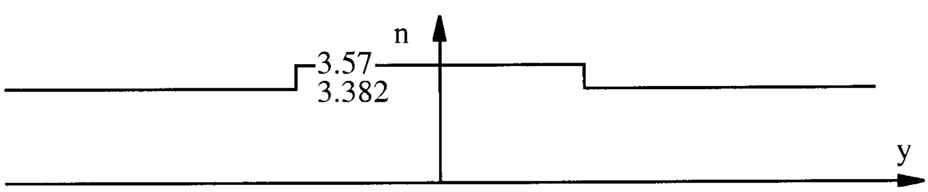
[2]

b) If the energy gap of $Ga_{1-x}Al_xAs$ varies $E_g = 1.42 + 1.24\,x$, the energy gap in the active layer (GaAs) is 1.42, while the energy gaps in the other layers of the hetero-structure ($Ga_{0.7}Al_{0.3}As$) are 1.792 eV. The band diagram is then as follows:



[4]

Similarly, if the refractive index of $Ga_{1-x}Al_xAs$ varies as $n = 3.57 - 0.6285\,x$, the index of the active layer is 3.57, while the indices of other the layers in the heterostructure are 3.382. The refractive index variation is then as shown below:

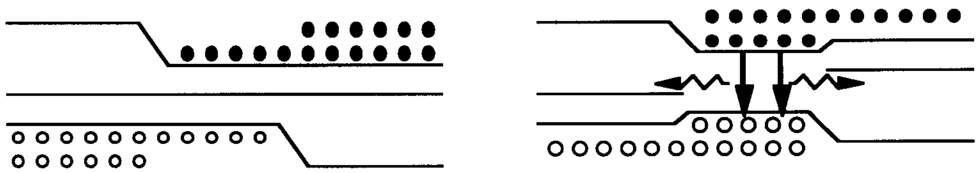


[2]

The emission wavelength can be found from the energy gap of the active region as:
$\lambda = hc/eE_g = (6.62 \times 10^{-34} \times 3 \times 10^{8}) / (1.6 \times 10^{-19} \times 1.42)$ m $= 0.87$ μm

[2]

c) Under forward bias, the P-side of the band diagram tilts down, while the N-side tilts up, as shown below.



Equilibrium   Forward bias

[2]

Electrons then move to the left, while holes move to the right. When the electrons reach the step in the conduction band at the P-n junction, they are prevented from moving further to the left. Holes are prevented from moving to the right when they reach the step in the valence band at the n-N junction. Consequently, there is a large electron density in the conduction band in the active region, and a large hole density in the valence band. These populations are inverted from the equilibrium distribution. A photon of the correct energy can then easily trigger the radiative transition required in stimulated emission. The photons are confined to the active region by the refractive index steps at the two heterojunctions.

[2]

6. The lumped-element rate equations for a semiconductor laser are:

$$dn/dt = I/ev - n/\tau_e - G\phi(n - n_0)$$
$$d\phi/dt = \beta n/\tau_{rr} + G\phi(n - n_0) - \phi/\tau_p$$

a) The individual terms in these equations are:

carrier injection	I/ev
recombination	$-n/\tau_e$
spontaneous emission	$\beta n/\tau_{rr}$
absorption and stimulated emission	$G\phi(n - n_0)$
radiation from the cavity	$-\phi/\tau_p$

[3]

The parameter n_0 is the electron density at which the rates of absorption and stimulated emission just balance, i.e. the electron density at which the material becomes transparent.

[1]

b) Below threshold, stimulated emission may be neglected, so the steady state equations reduce to:

$$I/ev - n/\tau_e = 0$$
$$\beta n/\tau_{rr} - \phi/\tau_p = 0$$

[2]

Above threshold, spontaneous emission may be neglected, so the steady state equations reduce to:

$$I/ev - n/\tau_e - G\phi(n - n_0) = 0$$
$$G\phi(n - n_0) - \phi/\tau_p = 0 \quad \text{or} \quad G(n - n_0) - 1/\tau_p = 0$$

[2]

From the upper equations, the electron density below threshold is $n = I\tau_e/ev$

[2]

From the lower ones, the electron density above threshold is $n = n_0 + 1/G\tau_p = n_{th}$

[2]

The threshold current I_{th} is reached when the electron density rises to n_{th}, so that $I_{th} = n_{th}ev/\tau_e$.

[1]

The threshold current is most effectively reduced by minimising n_{th}. Since n_0 is largely fixed, the gain constant G should be maximised by ensuring that the optical field has a large overlap with the active region. A large photon lifetime is also useful; this can be achieved using a long cavity with highly reflective mirrors.

[1]

c) i) Below threshold, n varies linearly with I, with a slope $dn/dI = \tau_e/ev$. Above threshold, n is held constant at the value n_{th}.

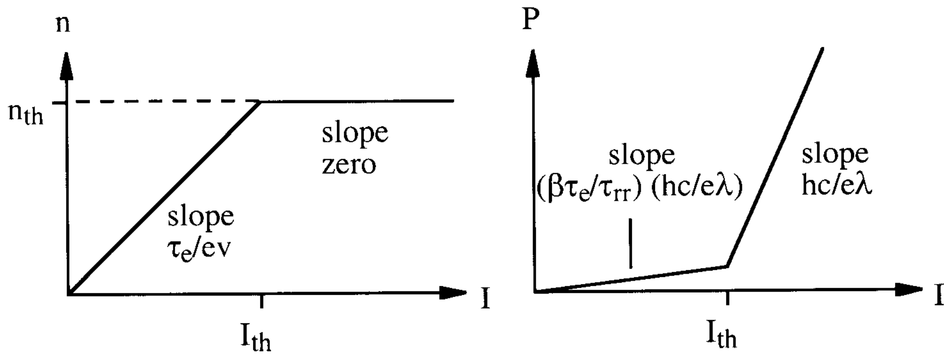
ii) The rate of change of photon density due to radiation out of the cavity is $-\phi/\tau_p$. The total output photon flux is therefore $\Phi = v\phi/\tau_p$. Each photon carries energy hc/λ, so the output power is $P = (hc/\lambda)(v\phi/\tau_p)$.

Below threshold, $\phi/\tau_p = \beta n/\tau_{rr} = \beta n/\tau_{rr} = \beta(\tau_e/\tau_{rr})$ (I/ev)
The output power therefore varies as $P = \beta\ (hc/e\lambda)\ (\tau_e/\tau_{rr})\ I$
P therefore varies linearly with I, with a slope $dP/dI = \beta\ (hc/e\lambda)\ (\tau_e/\tau_{rr})$

Above threshold, $\phi/\tau_p = G\phi(n - n_0) = I/ev - n/\tau_e = \{I - I_{th}\}/ev$
The power output therefore varies as $P = (hc/e\lambda)\ \{I - I_{th}\}$
P therefore varies linearly with I, with a slope $dP/dI = (hc/e\lambda)$

[2]

The variations of n and P are then as shown below.



[4]