E3.10
Ix 3.7
C 1.6

Mathematics for Signals and Systems

Exam of May 2003

SOLUTIONS

Question 1 (a) $\alpha = \frac{1}{\sqrt{5}}$, $e_2 = \alpha \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Note that α is uniquely determined, but there are many other choices for e_2 and e_3.

(b) We have

$$T^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1, \quad T^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e_2 .$$

Since T^{-1} must also be unitary, and since $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is an orthonormal basis in $\mathbb{C}^3$, $\left\{T^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, T^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, T^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ must also be an orthonormal basis in $\mathbb{C}^3$. The simplest choice is to take $T^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e_3$. Thus,

$$T^{-1} = \left[e_1 \middle| e_2 \middle| e_3 \right] = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that

$$T = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) If λ is an eigenvalue of T then for some eigenvector $x \in \mathbb{C}^3$ we have $Tx = \lambda x$. Since T is unitary, we have $\|Tx\| = \|x\|$ (for all $x \in \mathbb{C}^3$). Thus, for the eigenvector we have $\|\lambda x\| = \|x\|$, so that $|\lambda| = 1$. Hence, no eigenvalue of T is contained in $\mathcal{D}$.

(d) A vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ belongs to $M^\perp$ if and only if

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It is easy to see that this is equivalent to $x_1 = x_2 = 0$ (x_3 may be any number).

(e) The space $M^{\perp\perp}$ consists of all vectors of the form $x = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$, where $x_1, x_2 \in \mathbb{C}$.

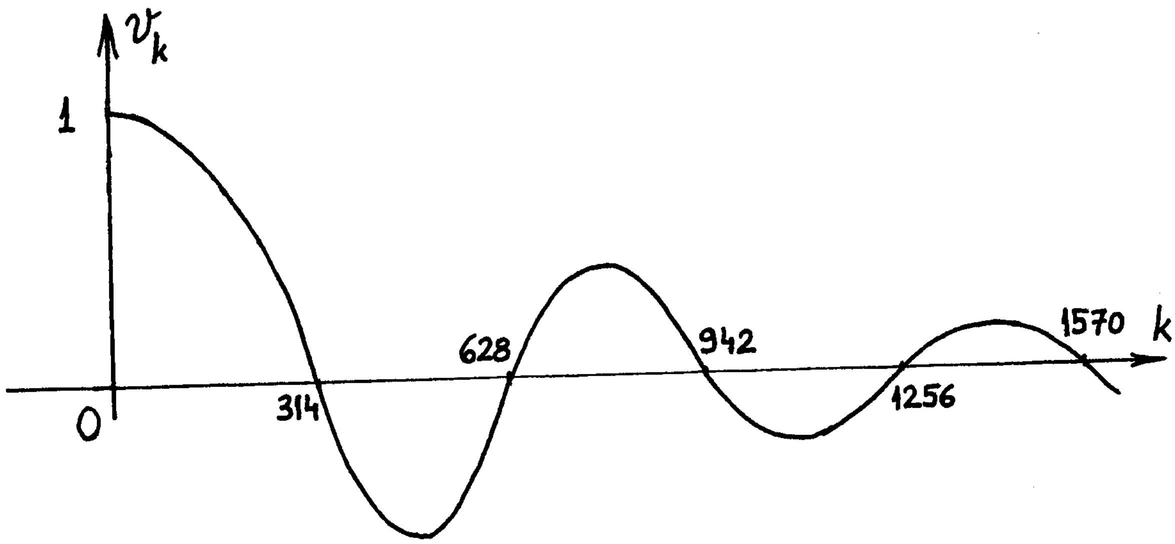
Hence, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. (So that Px retains only the first two components of x.)

(f) $P^2 = P$, $\quad \operatorname{rank} P = 2$, $\quad \|P\| = 1$.

Question 2 (a) $v \in \ell^2 \Rightarrow v \in c_0 \Rightarrow v \in \ell^\infty$.

v is not in ℓ^1, because for all values k such that $\sin(0.01k) \geq 0.5$ (and these k are situated in periodically recurring intervals) we have $v_k \geq \frac{50}{k}$, and $\left(\frac{1}{k}\right)$ is not in ℓ^1.

(b)



(c) Since $v \in \ell^2$, according to the Paley-Wiener theorem we have $\hat{v} \in H^2(\mathcal{E})$. Every function in $H^2(\mathcal{E})$ has boundary values almost everywhere on the unit circle, and the boundary function is in L^2, since

$$\|\hat{v}\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |\hat{v}(e^{i\varphi})|^2 \, d\varphi .$$

(We remark that, by the Paley-Wiener theorem, $\|v\|_2 = \|\hat{v}\|_2$.)

(d) The filter is time-invariant (and linear) and its transfer function is

$$F(z) = \frac{3 - 0.5z^{-1} - 2z^{-2}}{1 - 0.8z^{-1}} = \frac{3z^2 - 0.5z - 2}{z^2 - 0.8z} .$$

This F is proper (i.e., it has a finite limit as $z \longrightarrow \infty$) and its poles are

$$z_1 = 0\,, \qquad z_2 = 0.8\,.$$

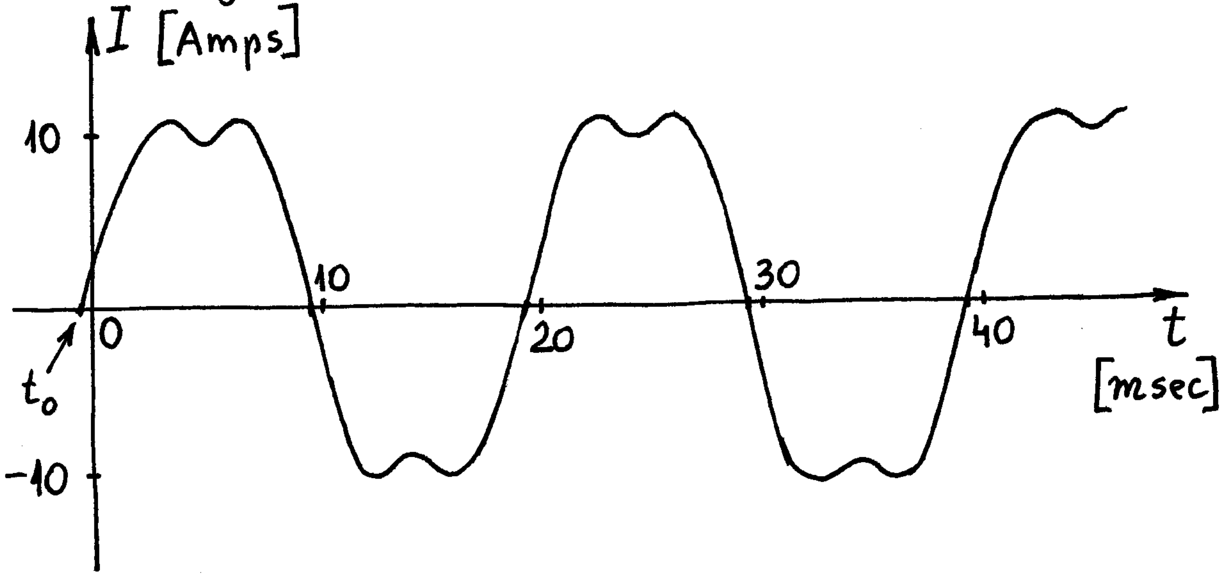
These poles are in $\mathcal{D}$, so that F is stable (i.e., $F \in H^\infty(\mathcal{E})$).

(e) The first four statements are true. Indeed, we have seen earlier that $v \in \ell^2$ and F is stable, so that $y \in \ell^2$. This implies $y \in c_0$ and this implies $y \in \ell^\infty$. By the Paley-Wiener theorem (discrete-time version), $y \in \ell^2$ implies $\hat{y} \in H^2(\mathcal{E})$.

If we define $\hat{y}$ also in $\mathcal{D}$ via $\hat{y} = F\hat{v}$ (both $\hat{v}$ and F can be defined on $\mathcal{D}$ by analytic continuation, except at a finite number of poles) then $\hat{y}$ will have poles at the poles of F (computed at part (d)), which are in $\mathcal{D}$. Hence, $\hat{y} \in H^2(\mathcal{D})$ cannot be true.

Question 3

(a) The period is $T = 20$ msec (corresponding to 50 Hz). There is a fundamental component of frequency 50 Hz and amplitude 10, a third harmonic (150 Hz) of amplitude 2, and a seventh harmonic (350 Hz) of amplitude 0.1. This seventh harmonic is so small that it can be neglected in the plot. The three components of I are synchronized in the sense that they cross zero simultaneously at $t_0 = -\frac{1}{1000\pi}$. Sketching the fundamental component, the third harmonic, and adding them, we obtain approximately:



(b) On $L^2[0,T]$ we define the inner product $\langle f, g\rangle = \frac{1}{T}\int_0^T f(t)\,\overline{g(t)}\,dt$, and we put $\|f\|^2 = \langle f, f\rangle$. Then $I_{RMS}^2 = \|I\|^2$. Denote

$$e_k(t) = \sin k\, 100\pi (t - t_0), \quad t_0 = \frac{-1}{1000\pi}$$

$(k = 1, 3, 7)$.

From the computations done in the theory of Fourier series we know that e_1, e_3, e_7 are orthogonal and $\|e_1\|^2 = \|e_3\|^2 = \|e_7\|^2 = 1/2$. Hence $\|I\|^2 = \|10e_1 + 2e_3 + 0.1e_7\|^2 =$

$$= \left(10^2 + 2^2 + (0.1)^2\right) \cdot \frac{1}{2} = 52.005,$$

so that $I_{RMS} = \sqrt{52.005} \approx 7.21$ (Amps).

(c) $P = \langle U, I \rangle = \langle U, 10e_1 \rangle$ (by orthogonality)

$$= 3250 \cdot \langle \sin 100\pi t, \sin(100\pi t + 0.1) \rangle$$

$$= 3250 \cdot \frac{1}{2} \cos 0.1 \approx 1617 \quad \text{(Watts)}.$$

(d) I is band-limited in the sense that its Fourier transform $\mathcal{F}I$ vanishes for $\omega > 700\pi$. However, I is not in $L^2(-\infty, \infty)$, hence it is not contained in any of the spaces $BL(\omega_b)$ which appear in the sampling theorem.

(e) Yes, obviously.

(f) If we choose $t_0 = \frac{-1}{1000\pi}$ (see part (a)), then I_0 is continuous. However, I_0 cannot be band-limited, because it is not analytic. Indeed, either I_0 or its derivative is not continuous at t_0, for any choice of t_0.

Question 4 (a) $v \in H^2(\mathbb{C}_+)$, the others are not. Indeed, $|\theta(i\omega)| = |q(i\omega)| = 1$ for all $\omega \in \mathbb{R}$, so $\int_{-\infty}^{\infty} |\theta(i\omega)|^2 d\omega = \int_{-\infty}^{\infty} |q(i\omega)|^2 d\omega = \infty$. The remaining functions h and ψ have unstable poles.

(b) $v \in H^\infty(\mathbb{C}_+)$, $\|v\|_\infty = \frac{1}{2}$ $\left(\begin{smallmatrix}\text{obtained for} \\ s \to 0\end{smallmatrix}\right)$,

$\theta \in H^\infty(\mathbb{C}_+)$, $\|\theta\|_\infty = 1$ (see part (a)),

$q \in H^\infty(\mathbb{C}_+)$, $\|q\|_\infty = 1$ (see part (a)),

h and ψ are not in $H^\infty(\mathbb{C}_+)$, because of their unstable poles.

(c) θ and q determine isometric input-output operators. Indeed, if $\hat{y}(s) = \theta(s)\,\hat{u}(s)$ then from $|\theta(i\omega)| = 1$ (for all $\omega \in \mathbb{R}$) it follows that $\int_{-\infty}^{\infty} |\hat{y}(i\omega)|^2 d\omega = \int_{-\infty}^{\infty} |\hat{u}(i\omega)|^2 d\omega$.

By the Paley-Wiener theorem (continuous time version) it follows that $\|y\|_2 = \|u\|_2$. For q, the reasoning is the same.

(d) q, h are analytic on $\mathbb{C}_-$. The others have a pole in $\mathbb{C}_-$.

(e) $\mathcal{L}^{-1}(v)(t) = e^{-2t}$,

$$\mathcal{L}^{-1}(\theta)(t) = \delta_0(t) - 10\, e^{-5t} \quad (\delta_0 = \text{unit pulse})$$

$$\mathcal{L}^{-1}(q)(t) = \delta_0(t-4) \quad (\text{this is a delayed unit pulse})$$

$$\mathcal{L}^{-1}(h)(t) = \sin(t-1) \text{ for } t \geq 1, \ 0 \text{ else}$$

$$\mathcal{L}^{-1}(\psi)(t) = \frac{1}{2} e^{-2t} + \frac{1}{2} e^{2t} \quad \left(\begin{array}{c}\text{this follows from} \\ \text{the decomposition} \\ \text{below}\end{array}\right)$$

(f) $$\psi(s) = \frac{s}{s^2-4} = \underbrace{\frac{0.5}{s-2}}_{\psi_-} + \underbrace{\frac{0.5}{s+2}}_{\psi_+},$$

$\psi_- \in H^2(\mathbb{C}_-)$ and $\psi_+ \in H^2(\mathbb{C}_+)$.

(g) $$J = \int_{-\infty}^{\infty} \psi(i\omega)\overline{v(i\omega)}\, d\omega = 2\pi \langle \psi, v \rangle$$

$$= 2\pi \langle \psi_-, v \rangle + 2\pi \langle \psi_+, v \rangle$$

(we use the inner product of $L^2(i\mathbb{R})$).
Since the boundary functions of functions in $H^2(\mathbb{C}_-)$ and $H^2(\mathbb{C}_+)$ are orthogonal, the first term is zero. Thus,

$$J = 2\pi \langle \psi_+, v \rangle = \pi \langle v, v \rangle = \pi \|v\|^2.$$

We have seen in part (e) that $v = \mathcal{L}(a)$, where $a(t) = e^{-2t}$ ($a \in L^2[0,\infty)$). By the Paley-Wiener theorem, $\|v\| = \|a\| = \frac{1}{2}$, so that $J = \frac{\pi}{4}$.

$\boxed{\text{Question 5}}$ (a) Denoting the 2×2 matrix by A, the characteristic polynomial of A is

$$p(s) = \det(sI - A) = s^2 + \beta^2 s + 90{,}000.$$

A is stable iff the coefficients of A are positive. Thus, the system is stable iff $\beta \neq 0$.

(b) We have
$$(sI-A)^{-1} = \frac{1}{p(s)}\begin{bmatrix} s+\beta^2 & -300 \\ 300 & s \end{bmatrix},$$
whence
$$G(s) = [0 \;\; -\beta](sI-A)^{-1}\begin{bmatrix} 0 \\ \beta \end{bmatrix} = [0 \;\; -\beta]\frac{\beta}{p(s)}\begin{bmatrix} -300 \\ s \end{bmatrix}$$

$$= -\frac{\beta^2 s}{p(s)} = \frac{-0.01s}{s^2 + 0.01s + 90{,}000}.$$

If we examine $|G(i\omega)|$ for $\omega > 0$ (for $\omega < 0$ it is the same), we see that it tends to zero for $\omega \to 0$ or $\omega \to \infty$, and it has a peak for $\omega = 300$ (this can be seen also by drawing the Bode amplitude plot of G). To obtain the peak value, we substitute $\omega = 300$, which yields $G(300i) = -1$. Thus, $\|G\|_\infty = 1$ (precisely).

(c) We have $Tu = \mathcal{F}^{-1} G \mathcal{F} u$, $u \in L^2(-\infty,\infty)$, where $\mathcal{F}$ is the Fourier transformation. T is time-invar. and causal. Causality means that if $u \in L^2[t_0,\infty)$ for some $t_0 \in \mathbb{R}$, then also $Tu \in L^2[t_0,\infty)$ (i.e., $(Tu)(t) = 0$ for $t < t_0$). Time-invariance means that for any $t_0 \in \mathbb{R}$, $\quad S_{t_0} T S_{-t_0} = T$, where S_{t_0} is the right shift by t_0 on $L^2(-\infty,\infty)$.

Since S_{t_0} is unitary and it maps $L^2[0,\infty)$ onto $L^2[t_0,\infty)$, the formula on the bottom of the previous page implies that the norm of T on any of the spaces $L^2[t_0,\infty)$ is the same. According to the Fourés-Segal theorem, on $L^2[0,\infty)$, the norm of T is $\|G\|_\infty$ (which, in our specific case, is 1). Taking limits as $t_0 \to -\infty$, we obtain that $\|T\| = \|G\|_\infty = 1$.

(d) From $Tu = \mathcal{F}^{-1} G \mathcal{F} u$ we see that if $(\mathcal{F}u)(i\omega) = 0$ for $|\omega| > 100$, then also $(\mathcal{F}Tu)(i\omega) = 0$ for $|\omega| > 100$. Moreover, if $u \in L^2(-\infty,\infty)$, then also $Tu \in L^2(-\infty,\infty)$, since G is bounded on the imaginary axis $i\mathbb{R}$.

(e) For $\omega \in (0,100)$, $|G(i\omega)|$ is an increasing function. Thus, the maximal gain on the relevant frequency range is attained at $\omega = 100$. We have

$$G(100i) = \frac{-i}{-10{,}000 + i + 90{,}000}, \quad \text{so that}$$

$$|G(100i)| \approx \frac{1}{80{,}000} = 1.25 \cdot 10^{-5}, \quad \text{with}$$

a precision of $\pm 0.01\%$. Thus, the norm of T restricted to $BL(100)$ is $\approx 1.25 \cdot 10^{-5}$ (much less than its norm on $L^2(-\infty,\infty)$).