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AC4

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

MSc and EEE PART III: MEng, BEng and ACGI

Corrected Copy

INSTRUMENTATION

Thursday, 11 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions in this paper.

Answer FOUR questions

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s):	C. Papavassiliou
	Second Marker(s):	S. Lucyszyn

The Questions

1. Sensors and actuators

a) What is a sensor? What is an actuator? List examples of a temperature sensor and a temperature actuator. Also give examples of an AC pressure sensor and an AC pressure actuator.

[5]

b) Define the absolute linearity of a transducer. A transducer is given that converts an input x to an output y according to the following relationship:

$$y = 10x + x^2 \ , \ 0 < x < 1$$

i) Compute the best linear fit to the response of this transducer. (A linear fit is of the form $y = ax$)

[5]

ii) Compute the maximum discrepancy between the transducer response and the best linear fit you computed.

[2]

iii) What is the percentage integral non-linearity? To how many bits does it make sense to quantise the output of this transducer? Explain your answer.

[3]

iv) In a DC application you need to use a sensor described by the response range and equations of this question. Show how AC noise will corrupt the DC measurement giving an upwards bias to the readings, and calculate the magnitude of this bias.

[5]

2. Time measurement

a) A clock, at frequency f_c, is given as well as two digital dividers and a counter. Draw the block diagram of an instrument appropriate for measuring signal frequencies much smaller than the clock frequency.

[3]

b) Write two inequalities bracketing the signal frequency in terms of the clock frequency, the divider values and the counter indication after a complete cycle of operation of the instrument. Derive an expression for the most likely test frequency given the counter reading and the divider values. Treat the error in the measurement as a uniformly distributed random variable. (HINT: Assume that the final counter reading N is big, so that $N + a \approx N$ in the final expressions.)

[5]

c) Derive an expression for the mean error in the frequency measurement, by evaluating the difference between the upper and lower bounds of the actual frequency given the parameters of a measurement. Show that the measurement error is minimised if the reference oscillator is not prescaled. If a minimum reading refresh rate f_R is required, how does the measurement error scale with the refresh rate? (HINT: the counter reading for any test frequency is inversely proportional to the refresh rate)

[7]

d) The measurement you described in parts (a) and (b) is to be used in some application that requires the reading to be updated 10 times/s, and must be as accurate as possible. Show that, regardless of the test frequency, it is better to take one measurement lasting 100ms by appropriately dividing the test signal frequency, than to take as many readings as fit in 100 ms on the signal frequency without dividing it.

[5]

3. Electronic Noise. You may assume throughout that source and load impedances are all real and have a value of 50 ohms, and that everything is at a temperature of 293K.

a) Calculate the RMS noise voltage observed at the terminals of a 50 ohm resistor over a bandwidth of 10 GHz. [3]

b) Write a formula for the data capacity of a noisy channel in terms of its signal to noise ratio and its bandwidth. Calculate the minimum digital signal amplitude from a 10GHz noise bandwidth, 50 ohm source, which can support a 1Gb/s data rate. (The digital signal is a 50% duty cycle square wave which takes on the values of 0 and V volts)

[4]

c) Write a formula for the noise factor of a cascade of 4 components each of a given gain and noise factor. What is the noise factor of an attenuator having an attenuation factor A?

[4]

d) A receiver consists of a low noise amplifier of gain G = 10 dB and noise figure $F_A = 2$ dB followed by a passive filter of 3 dB loss, and a mixer of 6 dB conversion loss. After these stages there is an amplifier with 60dB gain and a noise figure F = 3 dB.

i) Calculate the total gain and noise figure of this receiver.

[5]

ii) Calculate the minimum input signal level that can support a 1 Gb/s data rate at the receiver output. The noise bandwidth is 10 GHz at the receiver input and 1 GHz at the receiver output. The signal bandwidth is 1 GHz both at the receiver input and at the receiver output.

[4]

Note : $k = 1.38 \times 10^{-23}$ [S.I.]

4. Phase locked loops

a) Draw a block diagram of a PLL and identify the different components. Allow a general component in the feedback path (i.e. anything, not just a frequency divider) Write an equation describing the phase transfer function in steady state operation of the PLL.

[5]

b) Explain why a loop filter is not necessary for the operation of a PLL. Show that in the absence of a loop filter the PLL functions as a 1^{st} order low pass filter. What is the low frequency gain of this filter? At what frequency is its pole? Interpret this result in terms of time domain signals (HINT: zero phase input is when the output is locked on a sinusoidal reference input)

[5]

c) Explain how a diode together with an appropriate filter may be used as a frequency multiplier, and also as a phase detector. Discuss the similarities and differences of using a PLL and a diode multiplier.

[5]

d) Design a PLL frequency divider. Why is such a circuit useful? Discuss any limitations a PLL frequency divider is subject to. Can you suggest a way to make a millimetre wave PLL by using a VCO at MHz frequencies?

[5]

5. Sampling and A/D conversion

a) Describe the operation of a chopper amplifier. Draw a block diagram of a chopper amplifier including any essential filters. Why are chopper amplifiers used instead of direct coupled amplifiers? What are the limitations imposed on the signal bandwidth?

[5]

b) Describe the operation of the Goldberg amplifier, and draw a diagram for it. What advantage does it offer compared to a chopper amplifier? Describe how the main concept of the Goldberg amplifier can be generalised so that a broadband amplifier can be made out of several narrowband amplifiers.

[5]

c) Derive an expression for the RMS quantisation noise energy of a signal quantised to $M = 2^N$ levels in terms of the signal amplitude and N. From this derive an expression for the maximum signal to quantisation noise ratio (SQNR) of a signal quantised to $M = 2^N$ levels. What is the spectral limitation on a signal sampled at a frequency f_S? How does the signal to quantisation noise ratio scale with sampling frequency for a fixed signal frequency?

[5]

d) Discuss a simple 12 bit oversampled A/D converter for a 20 kHz data bandwidth using a 16 level quantiser and an appropriate digital filter. The quantiser and the digital logic are capable of operating at as high a frequency as required. What is the minimum sampling frequency to meet the required specification? Suggest a way to reduce the required sampling frequency by 3 orders of magnitude.

[5]

6. Radio frequency measurements

a) Write an expression for the input impedance, at a frequency f of a cable of length L, characteristic impedance Z_0 and which supports waves at a frequency independent phase velocity c_0 when it is terminated at an impedance Z_T. If such a cable has a $Z_0 = 50\ \Omega$ and is terminated at the $Z_T = 1\ \mathrm{M\Omega}$, comment on the variation of its input impedance with frequency. What is the maximum and what the minimum **magnitude** of the input impedance? At what frequencies (expressed in terms of the cable length) do these occur?

[10]

b) All the cables in a laboratory have a characteristic impedance $Z_0 = 50\ \Omega$, and they have an index of refraction $\eta = 2$. The voltage gain of a high input impedance voltage amplifier needs to be tested at $f = 125 MHz$ using a low impedance signal generator and a high input impedance oscilloscope. Specify the shortest cables that can be used to connect the generator to the amplifier input and the amplifier output to the oscilloscope. The lengths must be such that this measurement can be performed accurately with no additional interpretation of the measurement, and with no loading of the generator or the amplifier. Please note that cables are available at lengths of $L = 50 + 10n\ \mathrm{cm}$. (n is an integer).

[5]

c) What would the required cable lengths be if the device under test was a current amplifier and the same equipment was used? What correction factor, if any, must be applied to the ratio of oscilloscope amplitude to source amplitude to obtain the current gain of the device under test (DUT)? (HINT: The input cable must transform the voltage source into a current source, and the output cable must make the oscilloscope input appear like a short circuit!) .

Once again cables can only be $L = 50 + 10n\ \mathrm{cm}$ long.

[5]

The Answers - 2006

ANSWERS 1:

a) **[bookwork]** Both sensors and actuators convert an input x to an output y. For sensors the output y is in the electrical domain, for actuators the input x is in the electrical domain. The sensor input can be anything, e.g. chemical, mechanical, electromagnetic. Conversely, the actuator output can be anything, including chemical, mechanical, electromagnetic. Examples of temperature sensors: Resistors, semiconductors, where the resistance varies with temperature, thermocouples. A heater or a peltier device are examples of temperature actuators. Microphones and loudspeakers are examples of AC pressure sensors and actuators.

[5]

b) **[application of theory]** Absolute linearity can be defined as the inverse of the maximum deviation of the response from a straight line passing through the origin, normalised to the full scale range of the instrument. For the example given the full scale range is y=10.1.

i) The maximum deviation from a straight line is a minimisation problem. First compute the best straight line:

$$\min \int_0^1 \left(10x + x^2 - ax\right)^2 dx \Rightarrow \frac{d}{da}\int_0^1 \left(10x + x^2 - ax\right)^2 dx = 0 \Rightarrow$$
$$\int_0^1 2x\left(10x + x^2 - ax\right) dx = 0 \Rightarrow \frac{10}{3} + \frac{1}{4} - \frac{a}{3} = 0 \Rightarrow$$
$$a = 10.75$$

[5]

ii) **[application of theory]**The maximum distance between

$$y = 10x + x^2 \ ,\ 0 < x < 1 \text{ and } y = 10.75x \text{ is}$$

$$\Delta y_{\max} = \left|\max\left(-0.75x + x^2\right)\right| = -.75x + x^2\Big|_{x=0.375} = 0.14063$$

which is $\frac{\Delta y_{\max}}{y_{\max}} = 0.01278 = \frac{1}{78.2}$ of full scale output.

[5]

iii) **[application of theory]** So we can say that the nonlinearity is 1.28%.

The linearity is better than 6 bits and worse than 7 bits, so it makes sense to quantise up to 7 bits.

iv) Apply the following signal to the sensor:

$$x = x_0 + \delta \sin(\omega t)$$

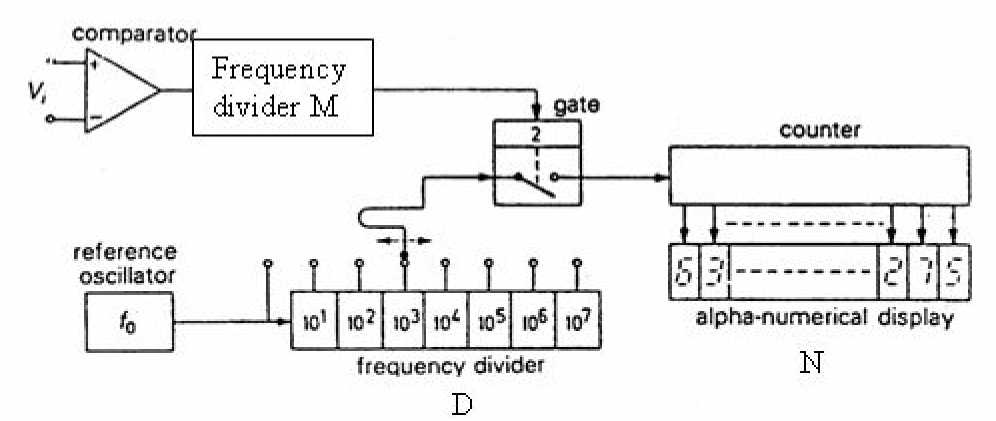
The DC component of the response is:

$$y = 10x_0 + 10\delta\sin(\omega t) + \left(x_0 + \delta\sin(\omega t)\right)^2 \Rightarrow$$
$$\overline{y} = 10x_0 + x_0^2 + \delta^2$$

So that the noise gives rise to an offset of magnitude $y_0 = \delta^2$ [5]

ANSWERS 2.

a) **[bookwork]**



[3]

b) **[bookwork]**

$$N = \text{int}\left(Mf_0 / fD\right) \Rightarrow \frac{Mf_0}{D\left(N+1\right)} < f < \frac{Mf_0}{DN}$$

The actual test frequency is clearly:

$$f = \frac{Mf_0}{D\left(N+x\right)} \approx \frac{Mf_0}{\text{DN}}\left(1-\frac{x}{N}\right),\ 0<x<1$$

Since x is uniformly distributed and 0<x<1, the mean value of this measurement is

$$f = \frac{Mf_0}{DN}\left(1-\frac{1}{2N}\right)$$

[5]

This reading differs from the naïve interpretation of $N \to N+1/2$ (also accepted as correct with -2 penalty) by:

$$\delta = \frac{Mf_0}{DN}\left(1-\frac{1}{2N}\right)-\frac{Mf_0}{D\left(N+1/2\right)} = \frac{Mf_0}{DN}\left(1-\frac{1}{2N}\right)-\frac{Mf_0}{DN\left(1+1/2N\right)} \approx$$
$$\frac{Mf_0}{DN}\left(1-\frac{1}{2N}-1+\frac{1}{2N}-O\left(\frac{1}{N^2}\right)\right) = \frac{Mf_0}{DN}\cdot O\left(N^{-2}\right)$$

The two estimates differ by a 3rd order correction in 1/N so they are not significantly different for large N.

c) **[new theory]**

A single reading is bound by the inequality:

$$\frac{Mf_0}{DN} < f < \frac{Mf_0}{D\left(N+1\right)} \Rightarrow \delta f = \frac{1}{2}\left(\frac{Mf_0}{DN}-\frac{Mf_0}{D\left(N+1\right)}\right) = \frac{1}{2}\left(\frac{Mf_0}{DN\left(N+1\right)}\right) \cong \frac{Mf_0}{2DN^2}$$

(δf is the mean uncertainty in the measurement).

However, the final count for a test frequency f_T is:

$N < \frac{f_0}{Df_R}$, so that the error is:

$$\delta f > \frac{Mf_0 D^2 f_R}{2Df_0^2} = \frac{MDf_R^2}{2f_0}$$

The minimum error is obtained with a prescaler value of D=1; the error scales as f_R^2

[7]

d) **[new theory]** For a refresh rate f_R, we must also satisfy $M < f_T / f_R = M_0$

We need to compare the error committed by a single measurement with $M = M_0$ against the error committed upon averaging M_0 measurements at $M = 1$.

the first error we have from part (b):

$$\delta f > \frac{M_0 Df_R^2}{2f_0} = \frac{f_T f_R}{2f_0}$$

The error of each M=1 measurement is:

$\delta f > \frac{f_R^2}{2f_0}$ by averaging these over $M < f_T / f_R = M_0$ readings, we reduce the error by a factor of $\sqrt{M_0} = \sqrt{f_T / f_R}$ so that the error in this scenario is bound by:

$\delta f > \frac{f_R^2}{2f_0}\sqrt{\frac{f_T}{f_R}} = \frac{f_T^{1/2} f_R^{3/2}}{2f_0}$. since $\frac{f_T}{f_R} > 1$, it follows it is preferable to take a single measurement at $M = M_0$.

[5]

ANSWER Question 3:

a) **[bookwork]** $v_n = \sqrt{4kTRB} = \sqrt{4 \cdot 1.38 \cdot 10^{-23} \cdot 293 \cdot 50 \cdot 10^{10}} = 89.9\mu\text{V}$

[3]

b) **[bookwork]** $C = B\log_2\left(\frac{S}{N}+1\right)$ where S is the signal power and N the noise power.

A square wave of amplitude V from an impedance Z has a power of :

$S = \frac{1}{2}\frac{V^2}{R}$, while the noise source has a power of $N = kTB = 4 \cdot 1.38 \cdot 10^{-23} \cdot 293 \cdot 10^{10} = 162\text{pW}$

since $B = 1GHz$ we need $\log_2\left(\frac{S}{N}+1\right) = 1 \Rightarrow \frac{S}{N} = 1 \Rightarrow V^2 = 8kTRB \Rightarrow V = 127\mu\text{V}$

[4]

c) **[computed example]** $F = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1G_2} + \frac{F_4 - 1}{G_1G_2G_3}$

(1 point for each term!)

[4]

d) **[computed example] A**n attenuator of attenuation factor A has a noise factor F=1/A

In the problem given, stages 1 and 4 are amplifiers and 2, 3 attenuators.

$F_1 = 2\text{dB} = 1.58$

$G_1 = 10\text{dB} = 10$

$F_2 = 1/G_2 = 3\text{dB} = 2$

$F_3 = 1/G_3 = 6\text{dB} = 4$

$F_4 = 3\text{dB} = 2$

We can now write:

$$F = F_1 + \frac{1/A_2 - 1}{G_1} + \frac{1/A_3 - 1}{G_1/A_2} + \frac{F_4}{G_1/\left(A_2A_3\right)} = 1.58 + \frac{1}{10} + \frac{3}{5} + \frac{1}{1.25} = 3.08$$

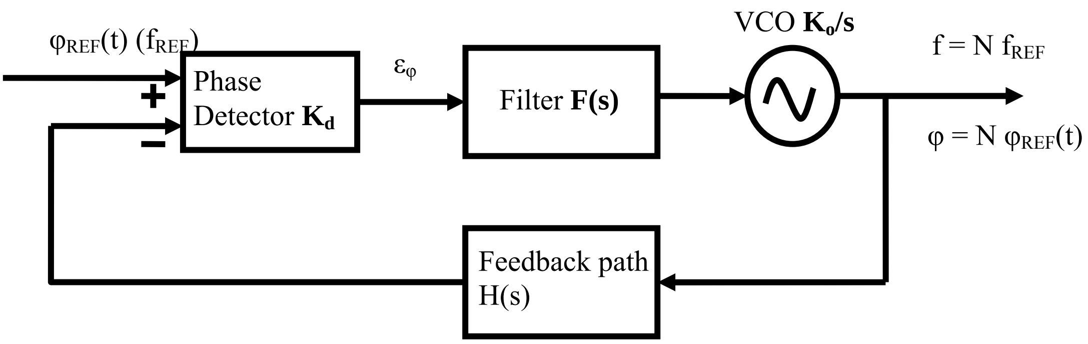
[5]

By definition, $F = \frac{SNR_{in}}{SNR_{out}}$. The required data rate will be supported if we increase the input power by a factor of 3.08, or the input signal voltage by a factor of $\sqrt{3.08} = 1.76$

[4]

ANSWER question 4:

a) **[bookwork]**



in steady state: $\frac{\varphi}{\varphi_{ref}} = \frac{K_d F K_o / s}{1 + K_d F K_o H / s}$

[5]

b) **[computed example]** There is already a pole (at f=0) in the transfer function of the VCO, so there IS a filter in the loop. Alternatively, the phase is the integral of the frequency the VCO produces, so there is an implicit integrator. Taking the previous transfer function, if F=1, $\frac{\varphi}{\varphi_{ref}} = \frac{N}{N / K_d K_0 s + 1} = \frac{H_0}{1 + s\tau_H}$

This has a low frequency gain of N and a pole at $\omega_p = K_d K_0 / N$.

This is a phase transfer function, which describes FM rejection about a reference input.

[5]

c) **[extension of theory]** A diode is a non-linear device, so that $i = f\left(v_1\right)$. The current can be sensed with a resistor. The diode transfer function can be written as a power series in the arguments. This suggests that the output will include power in all signal frequency harmonics, including DC. These output spectral components can be filtered and the required one can be separated. So a diode can be used as a frequency multiplier.

If the input signal is the sum of two signals, the output includes products of all powers of each of the inputs, including the product of the two signals. A phase detector is just a signal multiplier, so a diode can be used as a phase detector.

[5]

d) **[extension of theory]** use a diode as a phase detector and as a frequency multiplier in the feedback loop. In fact, a single diode is probably enough to do both tasks.

A PLL frequency divider is useful at frequencies where it is difficult to make counters , gates or analog multipliers.

An important limitation of a PLL frequency divider is the (phase) gain in the feedback path, which increases the loop gain, and consequently reduces the phase margin of the circuit.

An additional diode frequency multiplier after the VCO can be used to implement a high frequency PLL using UHF components.

[5]

ANSWERS 5:

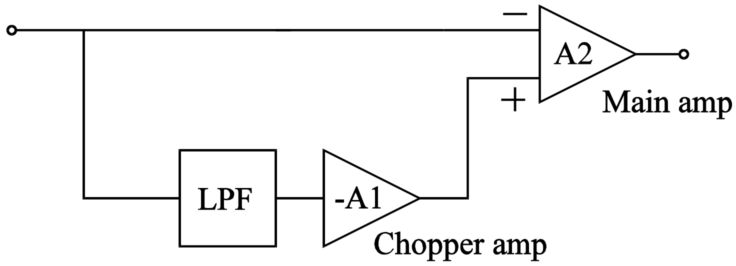
a) **[bookwork]** A chopper amplifier is : Low pass (antialiasing filter), chopper, High pass (or band pass) filter, amplifier, chopper , low pass filter.

It is effectively a sequence of an AM modulator and demodulator with signal conditioning filters.

It is used to obtain large gains at small signal frequencies. The signal bandwidth needs to be less than half the chopper frequency.

[5]

b) **[bookwork]** A Goldberg amplifier is a low-pass chopper amplifier in parallel with another amplifier.



The low-pass filter prevents high frequencies from reaching the chopper amplifier, and thus avoids aliasing. Low frequency input signals are fed into the drift-free chopper amplifier, and are thus amplified by $-\mathbf{A_1 A_2}$.

Assume LPF: $T_1(s) = \dfrac{1}{1+s\tau_1}$ Main amp: $T_2(s) = \dfrac{A2}{1+s\tau_2}$

Thus $V_{out} = A_2(V_+ - V_-) = \dfrac{A_2}{1+s\tau_2}\left(\dfrac{-A_1}{1+s\tau_1} - 1\right) \approx -\dfrac{A_2 A_1}{1+s\tau_2}\left(\dfrac{1+s\tau_1/A_1}{1+s\tau_1}\right)$

We can choose $\tau_2 \approx \tau_1 / A_1$, to maintain a dominant pole response.

The spectrum partitioning can be generalised to connect a number of bandpass amplifiers in parallel. This is essentially the approach of designing a high order filter.

[5]

c) **[bookwork]** The quantisation noise is assumed to be Gaussian and have a uniform PDF between ± ½ LSB, also denoted ± q/2, i.e.

$p(e_q) = 1/q \ (-q/2 < e_q < q/2)$

$p(e_q) = 0$ elsewhere

The ideal average (mean square) quantisation noise power is assumed to be white between $-f_s/2 < f < f_s/2$ and its total power is:

$$\bar{E}^2(e_q) = \int_{-\infty}^{\infty} e_q^{\,2} p(e_q) \ de_q \quad = \int_{-q/2}^{q/2} \frac{e_q^{\,2}}{q} \ de_q = \frac{q^2}{12}$$

Since the power of a sinusoidal signal of amplitude A is just $P = A^2/2 = 2^{2N-3} q^2$**,** given that $2A = 2^N q \Rightarrow A = 2^{N-1} q$, then

$$SQNR = 10\log\left(6A^2/q^2\right) = 10\log\left(3\cdot 2^{2N-1}\right) = \left(1.76 + 6.02N\right)\text{dB}$$

d) **[extension of theory + computed example]**By oversampling at f_{OSR} we leave the signal power invariant but spread the quantisation noise energy over a bandwidth $B = f_{OSR}$. The quantisation noise power is then:

$PSD\left(P_{QN}\right)=\frac{q^2}{12}/\,f_{OSR}$ while $PSD\left(P_S\right)=A^2\,/\,f_N$

So that $SNQR \propto 2^{2n-1} \propto \frac{f_{OSR}}{f_N}$ which means that to get 12 bits at 20kHz out of 4 bit oversampled conversion we need an oversampling ratio $OSR = 2^{16} = 65K$ and consequently a sampling frequency of $f_S = 40 \cdot 2^{16} = 2.6GHz$! A 1st order Delta modulator of the same specs, and the same quantiser only requires an oversampling ratio of $OSR = 2^{8/1.5} = 28.5$ and therefore a sampling frequency of $f_s = 40K \cdot 28.5 = 1.37MHz$

[5]

Answers, Q6

a) **[bookwork + extension]** $Z_{IN} = Z_0 \frac{Z_T + jZ_0 \tan\left(2\pi fL/c\right)}{Z_0 + jZ_T \tan\left(2\pi fL/c\right)}$, with $\frac{fL}{c}$ is the cable length measured in wavelengths.

If, as given, $Z_T > Z_0$, and the cable is so that $x = \tan\left(2\pi fL/c\right)$,

$$Z_T = Z_0 \frac{Z_T + jZ_0 x}{Z_0 + jZ_T x} = Z_0 \frac{1 + j\rho x}{\rho + jx} = Z_0 \frac{\left(1 + j\rho x\right)\left(\rho - jx\right)}{\rho^2 + x^2} = Z_0 \frac{\rho\left(1 + x^2\right) + jx\left(\rho^2 - 1\right)}{\rho^2 + x^2}, \rho = \frac{Z_0}{Z_T}$$

This will be capacitive for $x > 0 \Rightarrow n\frac{\lambda}{2} < L < n\frac{\lambda}{2} + \frac{\lambda}{4}$ and inductive for

$x < 0 \Rightarrow n\frac{\lambda}{2} + \frac{\lambda}{4} < L < \left(n+1\right)\frac{\lambda}{2}$

The magnitude of the input impedance is: $\left|Z_T\right| = Z_0 \left|\frac{Z_T + jZ_0 x}{Z_0 + jZ_T x}\right| = Z_0 \sqrt{\frac{Z_T^2 + x^2 Z_0^2}{Z_0^2 + x^2 Z_T^2}}$, $x = \tan\left(2\pi fL/c\right)$

To find the extrema of $\left|Z_T\right|$ we need to examine its derivative with respect to $x = \tan\left(2\pi fL/c\right)$:

$$\frac{d}{dx}\frac{Z_T^2 + x^2 Z_0^2}{Z_0^2 + x^2 Z_T^2} = 2xZ_0^2\left(Z_0^2 + x^2 Z_T^2\right) - 2xZ_T^2\left(Z_T^2 + x^2 Z_0^2\right) =$$
$$= 2x\left(Z_0^4 + x^2 Z_T^2 Z_0^2 - Z_T^4 - x^2 Z_T^2 Z_0^2\right) = 2x\left(Z_0^4 - Z_T^4\right)$$

if, as given, $Z_T > Z_0$, this has the opposite sign of $x = \tan\left(2\pi fL/c\right)$. The maxima are then attained at $x = 0 \Rightarrow L = \frac{n\lambda}{2}$ where $Z_{in} = Z_T = 1\ \text{M}\Omega$, and the minima at $x \to \infty \Rightarrow L = \frac{2n+1}{4}\lambda$, where $Z_{in} = \frac{Z_0^2}{Z_T} = 2.5\ \text{m}\Omega$. Clearly the minimum is less than the likely ohmic resistance of the cable. [10]

b) **[computed example]** Cables of length $L = n\frac{\lambda}{2}$ are unity impedance transformers. So both cables must be $L = \frac{\lambda}{2} = \frac{75}{f} = \frac{75}{1.25} = 60\ \text{cm}$ long. Note that the cables are transmission inverters, so that each contributes a phase of π to the gain. [5]

c) **[computed example]** The cables now need to be $L = \frac{\left(2n+1\right)\lambda}{4}$, with n integer. The shortest length which satisfies this relation is $\lambda = 30\ \text{cm}$, which is not available. then we need the next acceptable length, namely $\lambda = 90\ \text{cm}$.

We know that on a cable the voltage and current wave amplitudes are related by $\frac{v_0}{i_0} = Z_0$. Assuming the reflection coefficients to be unity, at the generator we have $v = 2v_0$, while at the DUT input we have $i = 2i_0$, so that the overall conversion is of magnitude unity. In transmission, though, we have half integer wavelengths, so there is an overall sign change between the instrument indication and the device gain.

[5]