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UNIVERSITY OF LONDON

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B.ENG. AND M.ENG. EXAMINATIONS 2003

For Internal Students of Imperial College

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART II : MATHEMATICS 4 (ELECTRICAL ENGINEERING)

Thursday 5th June 2003 2.00 - 4.00 pm

Answer FOUR questions.

Corrected Copy

[Before starting, please make sure that the paper is complete; there should be 4 pages, with a total of 6 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. Find the eigenvalues and normalised eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Using these, or otherwise, show that the matrix

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$$

diagonalises A such that

$$P^{-1}AP = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

2. Find the eigenvalues and normalised eigenvectors of the matrix

$$A = \begin{pmatrix} 11 & \sqrt{11} & 0 \\ \sqrt{11} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By writing the quadratic form

$$Q = 11x_1^2 + 2\sqrt{11}\, x_1 x_2 + x_2^2 + x_3^2$$

as

$$Q = \boldsymbol{x}^T A \boldsymbol{x},$$

where $\boldsymbol{x} = (x_1,\, x_2,\, x_3)^T$, show that Q can be written in the diagonal form

$$Q = 12y_1^2 + y_2^2,$$

by finding a matrix P which satisfies $\boldsymbol{x} = P\boldsymbol{y}$ where $\boldsymbol{y} = (y_1,\, y_2,\, y_3)^T$.

3. (i) The probability that an emitted particle will penetrate a certain shield is $p = 0.01$ and the particles act independently. If ten particles are emitted, what is the probability that

   (a) none penetrate the shield,

   (b) exactly one penetrates,

   (c) at least two penetrate?

   How many particles need to be emitted for the probability that at least one penetrates to be greater than $\frac{1}{2}$?

   (ii) Diagnostic tests A (chemical) and B (physical) for metal fatigue are available. The probability that A gives a correct diagnosis is p_A, i.e. $\mathrm{P}(E_A \,|\, mf) = p_A$ and $\mathrm{P}(\overline{E}_A \,|\overline{mf}) = p_A$, where $E_A = \{\textit{test A positive}\}$ and $mf = \{\textit{metal fatigue present}\}$; likewise, for test B, the probability of correct diagnosis is p_B. Further, the tests act independently in the sense that $\mathrm{P}(E_A \cap E_B \,|\, mf) = p_A p_B$ and $\mathrm{P}(E_A \cap E_B \,|\, \overline{mf}) = (1 - p_A)(1 - p_B)$. The proportion of metal samples that are fatigued is q.

   Calculate the probability that a metal sample for which both A and B give a positive result actually has metal fatigue. What is this probability if only A gives a positive result?

4. An electrical power system is subject to random fluctuations such that the voltage V at any instant has probability density $f(v) = \xi^{-1}(1 + v/\xi)^{-2}$ on $(0, \infty)$ with $\xi > 0$.

   Find the distribution function of V and calculate the median voltage. Evaluate $\mathrm{P}(V > a + b \,|\, V > a)$, where $0 < a < b$.

   Now suppose that the voltage is recorded at the same time on n successive days, producing independent readings $v_1, \ldots, v_n$.

   Calculate the probability that all n voltages lie in the range (a, b). For the case $n = 4$ and $\xi = 3$, calculate the probability that at most two of the readings fall below the level $a = 1$.

5. The random variable X has density function

$$f(x) = \frac{1}{2}\xi^3 x^2 e^{-\xi x} \text{ on } (0, \infty), \text{ with } \xi > 0\,.$$

Calculate $\mathrm{E}(X^{-1})$ and $\mathrm{var}(X^{-1})$, for which you may assume that

$$\int_0^\infty x^r e^{-\xi x}\,dx = r!/\xi^{r+1} \text{ for integer } r \geq 0\,.$$

A random sample $(x_1, \ldots, x_n)$ is obtained from the X-distribution. Show that the estimator $t = 2n^{-1}\sum_{i=1}^{n} x_i^{-1}$ is unbiased for ξ and compute its mean-square error. Is t consistent for ξ ?

6. An MA(3) process is defined as

$$y_t = e_t + \frac{1}{2}e_{t-1} + \frac{1}{4}e_{t-2}\,,$$

where $\{e_t\}$ is white noise with $\mathrm{E}(e_t) = 0$ and $\mathrm{var}(e_t) = \sigma_e^2$.

Evaluate $\mathrm{E}(y_t)$, $\mathrm{var}(y_t)$ and $\mathrm{cov}(y_t, y_{t-s})$ for $s \geq 1$. Is $\{y_t\}$ stationary?

Calculate the spectrum of $\{y_t\}$ and verify that $\{y_t\}$ forms a low-pass filtering of $\{e_t\}$.

END OF PAPER

M A T H E M A T I C S   D E P A R T M E N T

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{b}\cdot\mathbf{c}\times\mathbf{a} = \mathbf{c}\cdot\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\cdot\mathbf{a})\mathbf{b} - (\mathbf{b}\cdot\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary}, |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^{2n}}{(2n)!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \dots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n\frac{x^{n+1}}{(n+1)} + \dots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\,D^{n-1}g + \dots + \tbinom{n}{r} D^r f\,D^{n-r}g + \dots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \dots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!,\quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$