Master

UNIVERSITY OF LONDON

[E 2.9 (Maths 4) 2007]

B.ENG. AND M.ENG. EXAMINATIONS 2007

For Internal Students of Imperial College

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART II : MATHEMATICS 4 (ELECTRICAL ENGINEERING)

Thursday 31st May 2007 2.00 - 4.00 pm

Answer FOUR questions.

Please answer questions from Section A and Section B in separate answerbooks.

A statistics data sheet is provided.

[*Before starting, please make sure that the paper is complete; there should be 5 pages, with a total of 6 questions. Ask the invigilator for a replacement if your copy is faulty.*]

SECTION A

1. Find the eigenvalues and normalised eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Using these, or otherwise, show that the matrix

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$$

diagonalises A such that

$$P^{-1}AP = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

2. The quadratic form

$$Q = 7x_1^2 + 2\sqrt{7}\,x_1x_2 + x_2^2 + x_3^2$$

can be written, in matrix notation, as

$$Q = \boldsymbol{x}^T A\,\boldsymbol{x},$$

where A is a 3×3 symmetric matrix and $\boldsymbol{x} = (x_1, x_2, x_3)^T$.

(i) Find the matrix A, and its eigenvalues λ_1, λ_2, and λ_3 and normalised eigenvectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$.

(ii) Now form the 3×3 matrix $P = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$. Show that it is orthogonal, i.e. $P^T = P^{-1}$.

(iii) If new variables $\boldsymbol{y} = (y_1, y_2, y_3)^T$ are chosen such that $\boldsymbol{x} = P\boldsymbol{y}$, show that Q can be written as

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2.$$

(iv) Find also the components y_i in terms of x_i.

PLEASE TURN OVER

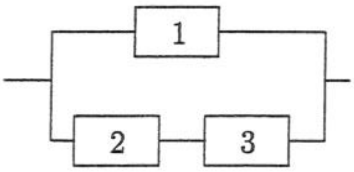
SECTION B

3. (i) Suppose that a system consists of k components in parallel, i.e. it functions as long as at least one of the components functions. Suppose that the states of the components are independent and that each component functions with probability $\frac{3}{4}$.

   (a) What is the probability that the system functions?

   (b) It is required that the system will fail with a probability of less than $1/1000$. How many components k need to be used?

   (ii) Consider a system in which there are three components with component 1 parallel to the serial components 2 and 3 as in the sketch.



   Suppose that the failure times of the components are independent and follow an exponential distribution with parameters $\lambda_1 = 3$, $\lambda_2 = 2$ and $\lambda_3 = 1$, respectively.

   (a) What is the probability that the system fails before a specified time t?

   (b) What is the median of the failure time distribution of the system?

4. (i) The discrete random variable X has probability mass function

$$p(x) = kx \quad \text{for} \quad x = 1, 2, 3, 4,$$

and $p(x)$ is zero for other values of x.

(a) Show that $k = \frac{1}{10}$.

(b) Calculate the mean and the variance of X.

(c) Find $\text{cov}(X, -5X)$.

(ii) The continuous random variable Y has the probability density function

$$f(y) = \begin{cases} \frac{3}{14}(y^2 + y) & \text{for } y \text{ in } (0,\ 2) \\ 0 & \text{for } y \text{ outside } (0,\ 2) \end{cases}$$

(a) Calculate the mean and the variance of Y.

(b) Find $\mathrm{E}[\{Y(1 + Y)\}^{-1}]$.

5. At Random University, students have to pass an oral examination. The examinations are spread out over 20 days. On any given day the examiner is one of the Professors A, B and C. Professor A examines on 5 days, Professor B on 8 days and Professor C on 7 days. It is known that 12% of the students fail when the examiner is Professor A, 11% if the examiner is Professor B and 10% if the examiner is Professor C. We observe one of the 20 days. Consider the event D that exactly one of the three candidates examined on that day fails.

(i) Compute the conditional probability of D given that Professor A was the examiner.

(ii) What is the probability of the event D?

(iii) What is the probability that Professor C was the examiner given that D occurred?

PLEASE TURN OVER

6. The random variables X_1 and X_2 are independent and each is uniformly distributed on $[0,\, \theta]$ for some $\theta > 0$.

   (i) Let $W = \max(X_1, X_2)$.

   (a) Find the cumulative distribution function F_W of W.

   (b) Show that for $x \in [0,\, \theta]$ the probability density function of W

   is $f_W(x) = \dfrac{2x}{\theta^2}$.

   (c) Find the expectation $E(W)$ and the variance Var (W) of W.

   (ii) Consider the two estimators

$$S = \frac{3}{2} \max(X_1,\, X_2) \quad \text{and} \quad T = X_1 + X_2 .$$

   (a) Show that both S and T are unbiased estimators for θ.

   (b) Find the variances of S and T. Which estimator is to be preferred?

   (iii) Find the maximum likelihood estimator of θ.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\, \mathbf{b},\, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary},\ |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\ ,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots\ ,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots\ ,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\ ;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1}\,Df\,D^{n-1}\,g + \ldots + \tbinom{n}{r}\,D^r f\,D^{n-r}g + \ldots + D^n f\ g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x,\, y)$ about $(a,\, b)$:

$$f(a+h,\, b+k) = f(a,\, b) + [hf_x + kf_y]_{a.b} + 1/2!\,\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a.b} + \ldots$$

(d) Partial differentiation of $f(x,\, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u,\, v)$, $y = y(u,\, v)$, then $f = F(u,\, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x,\, y)$ occur where $f_x = 0,\ f_y = 0$ simultaneously.
Let $(a,\, b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D > 0$ and $f_{xx}(a,\, b) < 0$, then $(a,\, b)$ is a maximum;
If $D > 0$ and $f_{xx}(a,\, b) > 0$, then $(a,\, b)$ is a minimum;
If $D < 0$ then $(a,\, b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\, y)dx + Q(x,\, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

1. Probabilities for events

For events A, B, and C $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\quad P(A)\,/\,P(\overline{A})$

Conditional probability $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\,P(B \mid A)\,P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) = \dfrac{P(A)\,P(B \mid A)}{P(A)\,P(B \mid A) + P(\overline{A})\,P(B \mid \overline{A})}$

A and B are independent if $\quad P(B \mid A) = P(B)$

A, B, and C are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable X is called the probability mass function (pmf) and is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$, so $\quad E(X^2) = \sum_x x^2 p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \mathrm{var}\,(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$

Sample variance $\quad s^2 = \dfrac{1}{n-1} \left\{ \sum_k x_k^2 - \dfrac{1}{n} \left(\sum_j x_j \right)^2 \right\}$ estimates σ^2

Standard deviation $\quad \mathrm{sd}\,(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness $\quad \beta_1 = E\left(\dfrac{X - \mu}{\sigma}\right)^3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X - \mu}{\sigma}\right)^4 - 3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\quad \widetilde{x}$ or x_{med}. Half the sample values are smaller and half larger

If the sample values $x_1, \ldots, x_n$ are ordered as $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$,
then $\widetilde{x} = x_{(\frac{n+1}{2})}$ if n is odd, and $\widetilde{x} = \frac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right)$ if n is even

α-quantile $Q(\alpha)$ is such that $P(X \leq Q(\alpha)) = \alpha$

Sample α-quantile $\widehat{Q}(\alpha)$ Proportion α of the data values are smaller

Lower quartile Q1 $= \widehat{Q}(0.25)$ one quarter are smaller

Upper quartile Q3 $= \widehat{Q}(0.75)$ three quarters are smaller

Sample median $\widetilde{x} = \widehat{Q}(0.5)$ estimates the population median $Q(0.5)$

3. **Probability distribution for a continuous random variable**

The cumulative distribution function (cdf) $\quad F(x) = P(X \leq x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The probability density function (pdf) $\quad f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x\,,\quad \mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2,$ where $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\mathrm{d}x$

4. **Discrete probability distributions**

Discrete Uniform *Uniform* (n)

$$p_x = \frac{1}{n} \quad (x = 1, 2, \ldots, n) \qquad \mu = (n+1)/2,\ \sigma^2 = (n^2-1)/12$$

Binomial distribution *Binomial* (n, θ)

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n) \qquad \mu = n\theta\,,\ \sigma^2 = n\theta(1-\theta)$$

Poisson distribution *Poisson* (λ)

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots) \quad (\text{with } \lambda > 0) \qquad \mu = \lambda,\ \sigma^2 = \lambda$$

Geometric distribution *Geometric* (θ)

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots) \qquad \mu = \frac{1}{\theta},\ \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. **Continuous probability distributions**

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \qquad \mu = (\alpha+\beta)/2\,,\ \sigma^2 = (\beta-\alpha)^2/12$$

Exponential distribution *Exponential* (λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \leq 0). \end{cases} \qquad \mu = 1/\lambda\,,\ \sigma^2 = 1/\lambda^2$$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \mathrm{var}(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t $\quad R(t) = P(T > t)$

The failure rate or hazard function $\quad h(t) = f(t)/R(t)$

The cumulative hazard function $\quad H(t) = \int_0^t h(t_0)\,\mathrm{d}t_0 = -\ln\{R(t)\}$

The Weibull(α, β) distribution has $\quad H(t) = \beta t^\alpha$

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P(\text{"device } i \text{ operates"})$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$

A system of devices in parallel operates if any device operates

$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1-R_1)(1-R_2)\cdots(1-R_k)$

8. Covariance and correlation

The covariance of X and Y $\quad \mathrm{cov}(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance $\quad s_{xy} = \dfrac{1}{n-1} S_{xy}$ estimates $\mathrm{cov}(X,Y)$

Correlation coefficient $\quad \rho = \mathrm{corr}(X,Y) = \dfrac{\mathrm{cov}(X,Y)}{\mathrm{sd}(X) \cdot \mathrm{sd}(Y)}$

Sample correlation coefficient $\quad r = \dfrac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$ estimates ρ

9. Sums of random variables

$$E(X+Y) = E(X)+E(Y)$$
$$\mathrm{var}\,(X+Y) = \mathrm{var}\,(X)+\mathrm{var}\,(Y)+2\,\mathrm{cov}\,(X,Y)$$
$$\mathrm{cov}\,(aX+bY,\ cX+dY) = (ac)\,\mathrm{var}\,(X)+(bd)\,\mathrm{var}\,(Y)+(ad+bc)\,\mathrm{cov}\,(X,Y)$$

If X is $N(\mu_1,\sigma_1^2)$, Y is $N(\mu_2,\sigma_2^2)$, and $\mathrm{cov}\,(X,Y)=c$, then $X+Y$ is $N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t $\quad \mathrm{bias}\,(t) = E(T)-\theta$

Standard error of t $\quad \mathrm{se}\,(t) = \mathrm{sd}\,(T)$

Mean square error of t $\quad \mathrm{MSE}(t) = E\{(T-\theta)^2\} = \{\mathrm{se}\,(t)\}^2+\{\mathrm{bias}\,(t)\}^2$

If $\overline{x}$ estimates μ, then $\mathrm{bias}\,(\overline{x})=0$, $\mathrm{se}\,(\overline{x})=\sigma/\sqrt{n}$, $\mathrm{MSE}(\overline{x})=\sigma^2/n$, $\widehat{\mathrm{se}}\,(\overline{x})=s/\sqrt{n}$

Central limit property $\quad$ If n is fairly large, $\overline{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.

For a random sample $x_1,x_2,\ldots,x_n$

$$\ell(\theta;\,x_1,x_2,\ldots,x_n) = P(X_1=x_1\,|\,\theta)\cdots P(X_n=x_n\,|\,\theta) \qquad \text{(discrete distribution)}$$
$$\ell(\theta;\,x_1,x_2,\ldots,x_n) = f(x_1\,|\,\theta)\,f(x_2\,|\,\theta)\cdots f(x_n\,|\,\theta) \qquad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum

12. Confidence intervals

If $x_1,x_2,\ldots,x_n$ are a random sample from $N(\mu,\sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\overline{x}-1.96\frac{\sigma}{\sqrt{n}},\ \overline{x}+1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0=t_{n-1,0.05}$

The 95% confidence interval for μ is $(\overline{x}-t_0\frac{s}{\sqrt{n}},\ \overline{x}+t_0\frac{s}{\sqrt{n}})$

13. Standard normal table — Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table — Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

m	p= 0.10	0.05	0.02	0.01	m	p= 0.10	0.05	0.02	0.01
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	∞	1.645	1.96	2.326	2.576

15. Chi-squared table — Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and p = .995, .975, *etc*

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$$p_{x\bullet} = \sum_y p_{xy} \quad \text{and} \quad P(X = x \,|\, Y = y) = \frac{p_{xy}}{p_{\bullet y}}$$

Continuous distribution

Joint cdf $\quad F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0\,, y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0$

Joint pdf $\quad f(x,y) = \dfrac{\mathrm{d}^2 F(x,y)}{\mathrm{d}x\,\mathrm{d}y}$

Marginal pdf of X $\quad f_X(x) = \int_{-\infty}^{\infty} f(x, y_0)\,\mathrm{d}y_0$

Conditional pdf of X given $Y = y$ $\quad f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is $\quad \widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}, \quad \widehat{\beta} = \dfrac{S_{xy}}{S_{xx}}$

The residual sum of squares $\quad \mathsf{RSS} = S_{yy} - \dfrac{S_{xy}^2}{S_{xx}}$

$\widehat{\sigma^2} = \dfrac{\mathsf{RSS}}{n-2} \qquad \dfrac{n-2}{\sigma^2}\,\widehat{\sigma^2}$ is from χ^2_{n-2}

$$E(\widehat{\alpha}) = \alpha, \quad E(\widehat{\beta}) = \beta,$$

$$\mathrm{var}\,(\widehat{\alpha}) = \frac{\sum x_i^2}{n\,S_{xx}}\sigma^2, \quad \mathrm{var}\,(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \mathrm{cov}\,(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}}\,\sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x, \qquad E(\widehat{y}_x) = \alpha + \beta x, \qquad \mathrm{var}\,(\widehat{y}_x) = \left\{ \frac{1}{n} + \frac{(x - \overline{x})^2}{S_{xx}} \right\} \sigma^2$$

$\dfrac{\widehat{\alpha} - \alpha}{\widehat{\mathrm{se}}\,(\widehat{\alpha})}, \quad \dfrac{\widehat{\beta} - \beta}{\widehat{\mathrm{se}}\,(\widehat{\beta})}, \quad \dfrac{\widehat{y}_x - \alpha - \beta x}{\widehat{\mathrm{se}}\,(\widehat{y}_x)}$ are each from t_{n-2}

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad \lambda_1 = 5 \quad \underline{a}_1 = (0,0,1)$$ 2

$$(\lambda-3)^2 - 1 = 0 \; ; \; \lambda^2 - 6\lambda + 8 = 0$$

$$\lambda_2 = 4, \; \underline{a}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \; ; \quad \lambda_3 = 2, \; \underline{a}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$ 3+3

Form
$$P = (\underline{a}_1\, \underline{a}_2\, \underline{a}_3) = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

There are 2 routes:

i) Direct calculation, which is acceptable.

ii) Prove in general that $P^T P = I$ based on the fact that A is symmetric so $\underline{a}_i \underline{a}_j = \delta_{ij}$ 12

$$P^T P = \begin{pmatrix} \underline{a}_1^T \\ \underline{a}_2^T \\ \underline{a}_3^T \end{pmatrix} (\underline{a}_1\, \underline{a}_2\, \underline{a}_3) = \{\underline{a}_i^T \underline{a}_j\} = I.$$

Moreover, $A\underline{a}_i = \lambda_i \underline{a}_i$

so $AP = P\Lambda \qquad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}$

$\therefore \quad P^{-1} A P = \Lambda,$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

i) $A = \begin{pmatrix} 7 & \sqrt{7} & 0 \\ \sqrt{7} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\lambda_1 = 8$ $\underline{a}_1 = (\sqrt{7}, 1, 0)^T/\sqrt{8}$

$\lambda_2 = 0$ $\underline{a}_2 = (1, -\sqrt{7}, 0)^T/\sqrt{8}$

$\lambda_3 = 1$ $\underline{a}_3 = (0, 0, 1)^T$ 6

ii) $P = (\underline{a}_1\ \underline{a}_2\ \underline{a}_3) = \frac{1}{\sqrt{8}}\begin{pmatrix} \sqrt{7} & 1 & 0 \\ 1 & -\sqrt{7} & 0 \\ 0 & 0 & \sqrt{8} \end{pmatrix}$

In general, if A is symmetric then $\underline{a}_i \cdot \underline{a}_j = \delta_{ij}$

$P^TP = \begin{pmatrix} \underline{a}_1^T \\ \underline{a}_2^T \\ \underline{a}_3^T \end{pmatrix}(\underline{a}_1\ \underline{a}_2\ \underline{a}_3) = \{\underline{a}_i^T\underline{a}_j\} = I \Rightarrow P^T = P^{-1}$. 6

Moreover $A\underline{a}_i = \lambda_i \underline{a}_i$ can be re-expressed as.

$AP = P\Lambda$ $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$

$\therefore$ $P^{-1}AP = P^TAP = \Lambda$.

iii) $\underline{x} = P\underline{y} \Rightarrow Q = \underline{x}^TA\underline{x} = \underline{y}^T(P^TAP)\underline{y}$

$= \underline{y}^T\Lambda y$ 4

Since $\Lambda = \mathrm{diag}(8, 0, 1)$

$Q = 8y_1^2 + y_3^2$

iv) $\underline{x} = P\underline{y} \Rightarrow \underline{y} = P^{-1}\underline{x} = P^T\underline{x}$

$\therefore$ $\underline{y} = \frac{1}{\sqrt{8}}\begin{pmatrix} \sqrt{7} & 1 & 0 \\ 1 & -\sqrt{7} & 0 \\ 0 & 0 & \sqrt{8} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ 4

$\sqrt{8}y_1 = \sqrt{7}x_1 + x_2$

$\sqrt{8}y_2 = x_1 - \sqrt{7}x_2$

$y_3 = x_3$

If the candidate uses a different ~~the~~ order of the eigenvalues and eigenvectors, then full marks should be given

SETTER: JDG CHECKER: ADG 17/11/06

3

EXAMINATION SOLUTIONS 2006-07

Course EE2(4)

Question 3

Marks & seen/unseen

Parts

(i) (a)

$$
\begin{aligned}
\mathrm{P}(\text{system functions}) &= 1 - \mathrm{P}(\text{all components fail}) \\
&= 1 - \prod_{i=1}^{k} \mathrm{P}(\text{component } i \text{ fails}) \\
&= 1 - \prod_{i=1}^{k} (1 - \mathrm{P}(\text{component } i \text{ functions})) \\
&= 1 - \prod_{i=1}^{k} \left(1 - \frac{3}{4}\right) = 1 - 4^{-k}
\end{aligned}
$$

seen similarly — 5

(b) Want: $\mathrm{P}(\text{system fails}) = 1 - \mathrm{P}(\text{system functions}) = 4^{-k} \leq \frac{1}{1000}$. — 2

This is equivalent to $4^k \geq 1000$. — unseen

Hence, $k \log_{10} 4 \geq 3$ and $k \geq 3/\log_{10} 4 \approx 4.98$. — 2

Since k needs to be a natural number, at least 5 components are needed. — 1

(check:$4^{-5} \approx 0.00098$)

(ii) (a) Let C_i denote the event "component i fails before time t". — 2

From the formula sheet: $\mathrm{P}(C_i) = 1 - \exp(-\lambda_i t)$.

Using the system structure and the independence we get

$$
\begin{aligned}
&\mathrm{P}(\text{system fails before time } t) = \mathrm{P}(C_1 \cap (C_2 \cup C_3)) \\
&= \mathrm{P}(C_1)\,\mathrm{P}(C_2 \cup C_3) \\
&= \mathrm{P}(C_1)(1 - \mathrm{P}(\overline{C_2} \cap \overline{C_3})) \\
&= \mathrm{P}(C_1)(1 - \mathrm{P}(\overline{C_2})\,\mathrm{P}(\overline{C_3})) \\
&= (1 - \exp(-3t))(1 - \exp(-2t)\exp(-1t)) \\
&= (1 - \exp(-3t))^2
\end{aligned}
$$

seen similarly — 4

(b) Need to solve $(1 - \exp(-3t))^2 = 0.5$ for t, where we know that $t \geq 0$.

This results in $t = -\frac{1}{3}\log(1 - \sqrt{0.5}) = \frac{1}{3}\log(2 + \sqrt{2})(\approx 0.409)$. — 4

Setter's initials Aly

Checker's initials RC

Page number 1

	EXAMINATION SOLUTIONS 2006-07	Course EE2(4)
Question 4		Marks & seen/unseen

Parts

(i) (a) k needs to be chosen such that $\sum_{i=1}^4 p(i) = 1$.
Hence, $1 = \sum_{i=1}^4 ki = k(1+2+3+4) = 10k$. Thus $k = \frac{1}{10}$. — 1, 1

(b) $\mathrm{E}(X) = \sum_{i=1}^4 ip(i) = \sum_{i=1}^4 i^2/10 = (1+4+9+16)/10 = 3$ — 2

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \sum_{i=1}^4 i^2 p(i) - 3^2 \\ &= \sum_{i=1}^4 i^3/10 - 9 = (1+8+27+64)/10 - 9 = 10 - 9 = 1\end{aligned}$$

Seen Similarly

4

(c) $\mathrm{cov}(X, -5X) = -5\,\mathrm{cov}(X, X) = -5\,\mathrm{Var}(X) = -5.$ — 3

(ii) (a)

$$\mathrm{E}(Y) = \int_{-\infty}^{\infty} yf(y)dy = \int_0^2 y\frac{3}{14}(y^2+y)dy = \frac{3}{14}\left[\frac{1}{4}y^4 + \frac{1}{3}y^3\right]_{y=0}^2$$
$$= \frac{3}{14}\left[\frac{16}{4} + \frac{8}{3}\right] = \frac{6}{7} + \frac{4}{7} = \frac{10}{7}(\approx 1.429).$$

3

Using

$$\mathrm{E}(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)dy = \int_0^2 y^2\frac{3}{14}(y^2+y)dy = \frac{3}{14}\left[\frac{1}{5}y^5 + \frac{1}{4}y^4\right]_{y=0}^2$$
$$= \frac{3}{14}\left[\frac{32}{5} + \frac{16}{4}\right] = \frac{48}{35} + \frac{6}{7} = \frac{78}{35}(\approx 2.229)$$

we get

$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{E}(Y^2) - (\mathrm{E}(Y))^2 = 78/35 - 100/49 \\ &= (546-500)/245 = 46/245(\approx 0.188).\end{aligned}$$

4

(b)

Unseen

$$\begin{aligned}\mathrm{E}[\{(Y(1+Y)\}^{-1}] &= \int_0^2 (y(1+y))^{-1} f(y)dy \\ &= \int_0^2 (y(1+y))^{-1}\frac{3}{14}y(1+y)dy = \int_0^2 \frac{3}{14}dy = \frac{3}{7}.\end{aligned}$$

2

Setter's initials	Checker's initials	Page number
AJ	RC	2

5

	EXAMINATION SOLUTIONS 2006-07	Course EE2(4)
Question 5		Marks & seen/unseen

Parts

(i) Let X denote the number of students that fail on this day. Conditional on Professor A being the examiner, X follows a Binomial distribution with parameters $n = 3$ and $p = 0.12$. Hence, letting A denote the event that Professor A is the examiner, we get
$\mathrm{P}(D|A) = \mathrm{P}(X = 1|A) = \binom{3}{1}p(1-p)^2 = 3 \cdot 0.12 \cdot 0.88^2 \approx 0.279.$ 6

(ii) By the law of total probability,

Seen Similarly

$$\begin{aligned}\mathrm{P}(D) &= \mathrm{P}(D|A)\,\mathrm{P}(A) + \mathrm{P}(D|B)\,\mathrm{P}(B) + \mathrm{P}(D|C)\,\mathrm{P}(C)\\ &= \mathrm{P}(D|A)\frac{5}{20} + \mathrm{P}(D|B)\frac{8}{20} + \mathrm{P}(D|C)\frac{7}{20},\end{aligned}$$

where B (resp. C) denotes the event that Professor B (resp. C) was the examiner.
Similar to the computation of $\mathrm{P}(D|A)$ we get
$\mathrm{P}(D|B) = 3 \cdot 0.11 \cdot 0.89^2 \approx 0.261$ and
$\mathrm{P}(D|C) = 3 \cdot 0.1 \cdot 0.9^2 \approx 0.243.$
Thus $\mathrm{P}(D) \approx 0.259.$ 8

(iii) Using Bayes rule,

$$\mathrm{P}(C|D) = \frac{\mathrm{P}(C \cap D)}{\mathrm{P}(D)} = \frac{\mathrm{P}(D|C)\,\mathrm{P}(C)}{\mathrm{P}(D)} \approx \frac{0.085}{0.259} \approx 0.328.$$

6

6

	EXAMINATION SOLUTIONS 2006-07	Course EE2(4)
Question 6		Marks & seen/unseen

Parts

(i) (a) By the independence of X_1 and X_2 for $0 \le x \le \theta$,

$$\begin{aligned} F_W(x) =& \mathrm{P}(W \le x) = \mathrm{P}(\max(X_1, X_2) \le x) \\ =& \mathrm{P}(X_1 \le x, X_2 \le x) = \mathrm{P}(X_1 \le x)\,\mathrm{P}(X_2 \le x) \\ =& \left(\frac{x}{\theta}\right)^2. \end{aligned}$$

seen similarly

Furthermore, $F_W(x) = 0$ for $x < 0$ and $F_W(x) = 1$ for $x > \theta$. 3

(b) Hence, for x in $[0, \theta]$, $f_W(x) = \frac{\partial}{\partial x} F_W(x) = \frac{2x}{\theta^2}$ and for all other values of x, $f_W(x) = 0$. 2

(It is irrelevant how $f_W(x)$ is set at $x = 0$ and $x = \theta$.)

(c) $\mathrm{E}(W) = \int_0^\theta x f_W(x) dx = \int_0^\theta x \frac{2x}{\theta^2} dx = \frac{2}{\theta^2}[\frac{1}{3}x^3]_{x=0}^\theta = \frac{2}{3}\theta$. 2

$\mathrm{E}(W^2) = \int_0^\theta x^2 f_W(x) dx = \int_0^\theta x^2 \frac{2x}{\theta^2} dx = \frac{2}{\theta^2}[\frac{1}{4}x^4]_{x=0}^\theta = \frac{2}{\theta^2}\frac{1}{4}\theta^4 = \frac{\theta^2}{2}$

and thus

$$\mathrm{Var}(W) = \mathrm{E}\left(W^2\right) - (\mathrm{E}(W))^2 = \frac{\theta^2}{2} - \left(\frac{2}{3}\theta\right)^2 = \frac{1}{18}\theta^2 (\approx 0.0556\theta^2).$$

2

(ii) (a) $\mathrm{E}(S) = \mathrm{E}(\frac{3}{2}W) = \frac{3}{2}\mathrm{E}(W) = \frac{3}{2}\frac{2}{3}\theta = \theta$ and 1

$\mathrm{E}(T) = \mathrm{E}(X_1 + X_2) = \mathrm{E}(X_1) + \mathrm{E}(X_2) = \frac{\theta}{2} + \frac{\theta}{2} = \theta$. 1

(b) $\mathrm{Var}(S) = \mathrm{Var}(\frac{3}{2}W) = \frac{9}{4}\mathrm{Var}(W) = \frac{9}{4}\frac{1}{18}\theta^2 = \frac{1}{8}\theta^2$. 1

Since X_1 and X_2 are independent $\mathrm{Var}(T) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2)$.

$$\begin{aligned} \mathrm{Var}(X_1) =& \mathrm{E}(X_1^2) - (\mathrm{E}(X_1))^2 = \int_0^\theta \frac{1}{\theta} x^2 dx - \left(\frac{\theta}{2}\right)^2 \\ =& [x^3/(3\theta)]_{x=0}^\theta - \theta^2/4 = \theta^3/(3\theta) - \theta^2/4 = \theta^2/12. \end{aligned}$$

As X_1 and X_2 are identically distributed, $\mathrm{Var}(X_2) = \mathrm{Var}(X_1)$.

Hence, $\mathrm{Var}(T) = 2\,\mathrm{Var}(X_1) = \theta^2/6$. 3

Both S and T are unbiased estimator of θ. As S has the smaller variance, S is to be preferred. 2

(iii) The density (likelihood) of the observations x_1, x_2 is $f_{X_1,X_2}(x_1, x_2) = \frac{1}{\theta^2}$ if both x_1 and x_2 are in $[0, \theta]$. Otherwise, $f_{X_1,X_2}(x_1, x_2) = 0$.

Unseen

Hence, f_{X_1,X_2} is maximised if θ is minimal subject to the constraint that $\theta \ge x_1$ and $\theta \ge x_2$.

Hence, the maximum likelihood estimator is $\max(X_1, X_2)$. 3

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