Master

UNIVERSITY OF LONDON

E2.9 Mathematics 4

B.ENG. AND M.ENG. EXAMINATIONS 2006

For Internal Students of Imperial College

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART II : MATHEMATICS 4 (ELECTRICAL ENGINEERING)

Wednesday 31st May 2006 2.00 - 4.00 pm

Answer FOUR questions.

Please answer questions from Section A and Section B in separate answerbooks.

A statistics data sheet is provided.

Corrected Copy

[Before starting, please make sure that the paper is complete; there should be 5 pages, with a total of 6 questions. Ask the invigilator for a replacement if your copy is faulty.]

SECTION A

1. Consider a real $n \times n$ symmetric matrix A with distinct eigenvalues λ_i and corresponding normalized eigenvectors $\boldsymbol{e}_i$ for $i = 1, \ldots n$.

   (a) Show that all the λ_i are real.

   (b) Show that the eigenvectors $\boldsymbol{e}_i$ obey the orthogonality relation

   $$\boldsymbol{e}_i^T \boldsymbol{e}_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

   (c) Show that the $n \times n$ matrix $P = \{\boldsymbol{e}_1 \, \boldsymbol{e}_2 \ldots \boldsymbol{e}_n\}$ satisfies the relation

   $$P^T P = I,$$

   where I is the $n \times n$ unit matrix.

2. Show that the quadratic form

   $$Q = 4x_1^2 - 4x_1x_2 + x_2^2 + 6x_3^2$$

   can be written as

   $$Q = \boldsymbol{x}^T A \boldsymbol{x},$$

   where $\boldsymbol{x} = (x_1, x_2, x_3)^T$ and A is a real symmetric matrix, which is to be found.

   Hence show that Q can be re-expressed in the diagonal form

   $$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2,$$

   where the λ_i are to be determined, by finding a matrix P that satisfies $\boldsymbol{x} = P\boldsymbol{y}$ where $\boldsymbol{y} = (y_1, y_2, y_3)^T$.

   Find y_1, y_2 and y_3 in terms of x_1, x_2 and x_3 from the matrix P.

PLEASE TURN OVER

SECTION B

3. (i) The discrete random variable X_1 takes values $0, 1$ and 2 with probabilities $1/2$, $1/3$ and $1/6$, respectively. A second random variable X_2 takes values 1 and 3 with probabilities $1/4$ and $3/4$, respectively, and is independent of X_1. Compute the probabilities:

   (a) $\mathrm{P}(X_1 + X_2 = 3)$

   (b) $\mathrm{P}(3X_1/X_2 < 2)$

   (ii) The probability that a job running on a CPU (Central Processor Unit) will fail has been estimated to be $q = 0.05$. Assume there is a cluster with $n = 10$ of these CPUs all working independently. What is the probability that:

   (a) no jobs fail?

   (b) exactly one job fails?

   (c) at least one job fails?

   (iii) The breakdowns of a piece of electrical equipment occur in a Poisson process of rate $\lambda = 0.1$ per year. Find the probability of:

   (a) one breakdown in 5 years;

   (b) no more than 2 breakdowns in one year.

4. (i) Let the positive random variable T represent the lifetime of an electrical component.

(a) Carefully define the hazard function $h(t)$ of the component.

Now suppose that the distribution of the lifetime is exponential, so that the probability density function is $f(t) = \lambda e^{-\lambda t}$, $t \geq 0$.

(b) Show that, for $s, t \geq 0$, T is memoryless, that is

$$\mathrm{P}(T > s + t \mid T > t) = \mathrm{P}(T > s) .$$

(c) Calculate the hazard function in this case, and interpret the answer.

(ii) Let the independent positive random variables T_A and T_B represent the lifetime of electrical components A and B, respectively.

(a) If $f_{T_A}(t)$ and $f_{T_B}(t)$ are the probability density functions for T_A and T_B, respectively, what is the joint probability density function $f_{T_A, T_B}(t_A, t_B)$ for T_A, T_B?

(b) Show that the probability that A fails before B is given by

$$\int_0^\infty F_{T_A}(t)\, f_{T_B}(t)\, dt .$$

(c) Assuming that T_A and T_B both have exponential failure time distributions, i.e.

$$f_{T_A}(t) = \begin{cases} \lambda_A e^{-\lambda_A t} & \text{if } t \geq 0 , \\ 0 & \text{otherwise} , \end{cases} \quad \text{and} \quad f_{T_B}(t) = \begin{cases} \lambda_B e^{-\lambda_B t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} , \end{cases}$$

with $\lambda_A, \lambda_B > 0$, find the odds ratio

$$\mathrm{P}\{\text{ A fails before B }\} / \mathrm{P}\{\text{ B fails before A }\} .$$

5. (i) A certain process for producing an industrial chemical yields a product containing two types of impurities. For a specified sample from this process, let Y_1 denote the proportion of impurities in the sample and Y_2 the proportion of type I impurity among all impurities found. Suppose that the joint distribution of Y_1 and Y_2 can be modelled by the following probability density function:

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 2(1-y_1) & 0 \le y_1 \le 1, \ 0 \le y_2 \le 1, \\ 0 & \text{otherwise}. \end{cases}$$

Find the expected value of the proportion of type I impurities in the sample.

(ii) Two brands of CPU processors, denoted A and B, are each guaranteed for one year. In a random sample of 50 CPUs of brand A, 12 were observed to fail before the guarantee period ended. A random sample of 60 brand B CPUs also revealed 12 failures during the guarantee period. Estimate the 98% confidence interval for the true difference between proportions of failures during the guarantee period.

6. (i) Let $X_1, \ldots, X_n$ be iid random variables from a Poisson distribution with unknown parameter λ. Assume $n \ge 2$.

(a) Show that $\Lambda_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\Lambda_2 = \frac{1}{2}(X_1 + X_2)$ are both unbiased estimators of λ.

(b) Which estimator is more efficient? Justify your answer.

(ii) An MA(3) process is defined as

$$y_t = e_t + \frac{1}{2}e_{t-1} + \frac{1}{4}e_{t-2},$$

where $\{e_t\}$ is the white noise with $\mathrm{E}(e_t) = 0$ and $\mathrm{var}(e_t) = \sigma_e^2$.

(a) Evaluate the expected value $\mathrm{E}(y_t)$ and variance $\mathrm{var}(y_t)$.

(b) Evaluate the covariance $\mathrm{cov}(y_t, y_{t-s})$ for $s \ge 1$.

(c) Is the process $\{y_t\}$ stationary? Justify your answer.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\ a_2,\ a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\ \mathbf{b},\ \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary},\ |x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\,,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots\,,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots\,,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1}\,Df\,D^{n-1}g + \ldots + \tbinom{n}{r}\,D^r f\,D^{n-r}g + \ldots + D^n f\;g\,.$$

(b) Taylor's expansion of $f(x)$ about $x=a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^nf^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!,\quad 0<\theta<1\,.$

(c) Taylor's expansion of $f(x,\ y)$ about $(a,\ b)$:

$$f(a+h,\ b+k) = f(a,\ b) + [hf_x + kf_y]_{a.b} + 1/2!\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a.b} + \ldots$$

(d) Partial differentiation of $f(x,\ y)$:

i. If $y=y(x)$, then $f=F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x=x(t)$, $y=y(t)$, then $f=F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x=x(u,\ v)$, $y=y(u,\ v)$, then $f=F(u,\ v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,,\qquad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\ y)$ occur where $f_x=0,\ f_y=0$ simultaneously.
Let $(a,\ b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D>0$ and $f_{xx}(a,\ b)<0$, then $(a,\ b)$ is a maximum;
If $D>0$ and $f_{xx}(a,\ b)>0$, then $(a,\ b)$ is a minimum;
If $D<0$ then $(a,\ b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\ y)dx + Q(x,\ y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)\,[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)\,[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2\ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

1. Probabilities for events

For events A, B, and C $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\quad P(A) / P(\overline{A})$

Conditional probability $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\, P(B \mid A)\, P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) = \dfrac{P(A)\, P(B \mid A)}{P(A)\, P(B \mid A) + P(\overline{A})\, P(B \mid \overline{A})}$

A and B are independent if $\quad P(B \mid A) = P(B)$

A, B, and C are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable X is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \text{var}\,(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$, where $E(X^2) = \sum_x x^2 p_x$

Sample variance $\quad s^2 = \dfrac{1}{n-1} \left\{ \sum_k x_k^2 - \dfrac{1}{n} \left(\sum_j x_j \right)^2 \right\}$ estimates σ^2

Standard deviation $\quad \text{sd}\,(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$

Skewness $\quad \beta_1 = E\left(\dfrac{X - \mu}{\sigma}\right)^3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X - \mu}{\sigma}\right)^4 - 3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\widetilde{x}$. If the sample values $x_1, \ldots, x_n$ are ordered $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$

$\widetilde{x} = x_{(\frac{n+1}{2})}$ if n is odd, and $\widetilde{x} = \frac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right)$ if n is even.

α-quantile $Q(\alpha)$ is such that $P(X \le Q(\alpha)) = \alpha$

Sample α-quantile $\widehat{Q}(\alpha)$ is the sample value for which the proportion of values $\le \widehat{Q}(\alpha)$ is α (using linear interpolation between values on either side)

The sample median $\widetilde{x}$ estimates the population median $Q(0.5)$.

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $\quad F(x) = P(X \le x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The probability density function (pdf) $\quad f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x\,, \quad \mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2,$$

where $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\mathrm{d}x$

4. Discrete probability distributions

Discrete Uniform *Uniform* (n)

$$p_x = \frac{1}{n} \quad (x = 1, 2, \ldots, n) \qquad \mu = \tfrac{1}{2}(n+1)\,, \ \sigma^2 = \tfrac{1}{12}(n^2-1)$$

Binomial distribution *Binomial* (n, θ)

$$p_x = \binom{n}{x}\theta^x(1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n) \qquad \mu = n\theta\,, \ \sigma^2 = n\theta(1-\theta)$$

Poisson distribution *Poisson* (λ)

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots) \ \ (\text{with } \lambda > 0) \qquad \mu = \lambda\,, \ \sigma^2 = \lambda$$

Geometric distribution *Geometric* (θ)

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots) \qquad \mu = \frac{1}{\theta}\,, \ \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. Continuous probability distributions

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta-\alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases} \qquad \mu = (\alpha+\beta)/2\,, \quad \sigma^2 = (\beta-\alpha)^2/12\,.$$

Exponential distribution *Exponential* (λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \qquad \mu = 1/\lambda\,, \\ 0 & (-\infty < x \le 0). \qquad \sigma^2 = 1/\lambda^2\,. \end{cases}$$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} \quad (-\infty < x < \infty)$$

$$E(X) = \mu\,, \quad \mathrm{var}\,(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t $\quad R(t) = P(T > t)$

The failure rate or hazard function $\quad h(t) = f(t)/R(t)$

The cumulative hazard $\quad H(t) = \int_0^t h(t_0)\,\mathrm{d}t_0 = -\ln\{R(t)\}$

The Weibull(α, β) distribution has $\quad H(t) = \beta t^\alpha$

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P($"device i operates"$)$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1-R_1)(1-R_2)\cdots(1-R_k)$$

8. Covariance and correlation

The covariance of X and Y $\quad \mathrm{cov}\,(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance $\quad s_{xy} = \frac{1}{n-1} S_{xy}$ estimates $\mathrm{cov}\,(X, Y)$

Correlation coefficient $\quad \rho = \mathrm{corr}\,(X, Y) = \frac{\mathrm{cov}\,(X, Y)}{\mathrm{sd}\,(X) \cdot \mathrm{sd}\,(Y)}$

Sample correlation coefficient $\quad r = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$ estimates ρ

9. Sums of random variables

$E(X + Y) = E(X) + E(Y)$

$\mathrm{var}\,(X + Y) = \mathrm{var}\,(X) + \mathrm{var}\,(Y) + 2\,\mathrm{cov}\,(X, Y)$

$\mathrm{cov}\,(aX + bY,\ cX + dY) = (ac)\,\mathrm{var}\,(X) + (bd)\,\mathrm{var}\,(Y) + (ad + bc)\,\mathrm{cov}\,(X, Y)$

If X is $N(\mu_1, \sigma_1^2)$, Y is $N(\mu_2, \sigma_2^2)$, and $\mathrm{cov}\,(X, Y) = c$,
then $X + Y$ is $N(\mu_1 + \mu_2,\ \sigma_1^2 + \sigma_2^2 + 2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t $\quad \mathrm{bias}(t) = E(T) - \theta$

Standard error of t $\quad \mathrm{se}\,(t) = \mathrm{sd}\,(T)$

Mean square error of t $\quad \mathrm{MSE}(t) = E\{(T - \theta)^2\} = \{\mathrm{se}\,(t)\}^2 + \{\mathrm{bias}(t)\}^2$

If $\overline{x}$ estimates μ, then $\mathrm{bias}(\overline{x}) = 0$, $\mathrm{se}\,(\overline{x}) = \sigma/\sqrt{n}$, $\mathrm{MSE}(\overline{x}) = \sigma^2/n$, $\widehat{\mathrm{se}}\,(\overline{x}) = s/\sqrt{n}$

Central limit property $\quad$ if n is fairly large, $\overline{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.
For a random sample $x_1, x_2, \ldots, x_n$

$$\ell(\theta;\, x_1, x_2, \ldots, x_n) = P(X_1 = x_1 \,|\, \theta) \cdots P(X_n = x_n \,|\, \theta) \quad \text{(discrete distribution)}$$

$$\ell(\theta;\, x_1, x_2, \ldots, x_n) = f(x_1 \,|\, \theta) f(x_2 \,|\, \theta) \cdots f(x_n \,|\, \theta) \quad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum.

12. Confidence intervals

If $x_1, x_2, \ldots, x_n$ are a random sample from $N(\mu, \sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\overline{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \overline{x} + 1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0 = t_{n-1,0.05}$

The 95% confidence interval for μ is $(\overline{x} - t_0\frac{s}{\sqrt{n}},\ \overline{x} + t_0\frac{s}{\sqrt{n}})$

13. Standard normal table

Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table

Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

m	$p=$ 0.10	0.05	0.02	0.01	m	$p=$ 0.10	0.05	0.02	0.01
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	∞	1.645	1.96	2.326	2.576

15. Chi-squared table

Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and $p = .995, .975,$ *etc*

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76,15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ.

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$p_{x\bullet} = \sum_y p_{xy}$, and $P(X = x \mid Y = y) = \frac{p_{xy}}{p_{\bullet y}}$

Continuous distribution

Joint cdf $F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0, y_0)\, dx_0\, dy_0$

Joint pdf $f(x,y) = \dfrac{d^2 F(x,y)}{dx\, dy}$

Marginal pdf of X $f_X(x) = \int_{-\infty}^{\infty} f(x, y_0)\, dy_0$

Conditional pdf of X given $Y = y$ $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is

$$\widehat{\alpha} = \overline{y} - \overline{x}\widehat{\beta}, \quad \widehat{\beta} = S_{xy}/S_{xx}$$

The residual sum of squares RSS $= S_{yy} - \dfrac{S_{xy}^2}{S_{xx}}$

$$\widehat{\sigma^2} = \frac{\text{RSS}}{n-2}, \qquad \frac{n-2}{\sigma^2}\,\widehat{\sigma^2} \text{ is from } \chi^2_{n-2}$$

$$E(\widehat{\alpha}) = \alpha, \quad E(\widehat{\beta}) = \beta,$$

$$\text{var}(\widehat{\alpha}) = \frac{\sum x_i^2}{n\, S_{xx}}\sigma^2, \quad \text{var}(\widehat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \text{cov}(\widehat{\alpha}, \widehat{\beta}) = -\frac{\overline{x}}{S_{xx}}\sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x, \qquad E(\widehat{y}_x) = \alpha + \beta x, \qquad \text{var}(\widehat{y}_x) = \left\{\frac{1}{n} + \frac{(x - \overline{x})^2}{S_{xx}}\right\}\sigma^2$$

$$\frac{\widehat{\alpha} - \alpha}{\widehat{\text{se}}(\widehat{\alpha})}, \quad \frac{\widehat{\beta} - \beta}{\widehat{\text{se}}(\widehat{\beta})}, \quad \frac{\widehat{y}_x - \alpha - \beta x}{\widehat{\text{se}}(\widehat{y}_x)} \quad \text{are each from } t_{n-2}$$

19. Design matrix for factorial experiments With 3 factors each at 2 levels

$$\mathbf{X} = \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course: Paper 4

Question: EE(4)1

Marks & seen/unseen: Seen

Parts

$$A\underline{e}_i = \lambda_i \underline{e}_i \; ; \; \underline{e}_i^{*T} e_i = 1 \qquad i = 1, \dots, n$$

a) $\underline{e}_i^{*T} A \underline{e}_i = \lambda_i \underline{e}_i^{*T} \underline{e}_i$ by LH multiplying by $\underline{e}_i^{*T}$

Alternatively, * & transpose & RH multiply by $\underline{e}_i$

$$\underline{e}_i^{*T} A^{*T} = \lambda_i^* \underline{e}_i^{*T} \rightarrow \underline{e}_i^{*T} A^{*T} \underline{e}_i = \lambda_i^* \underline{e}_i^{*T} \underline{e}_i$$ 6

If A is a real symm mx, we have $A^{*T} = A^T = A$:

thus $\lambda_i = \lambda_i^* \rightarrow$ so λ_i are real.

b) Now we have λ_i real (& thus $\underline{e}_i$ real), write

$\underline{e}_i^T A e_j = \lambda_j \underline{e}_i^T \underline{e}_j$; $\quad \underline{e}_i^T A^T = \lambda_i \underline{e}_i^T$

where we've LH multiplied by $\underline{e}_i^T$ — RH multiply by $\underline{e}_j$ 7

subtract $\underline{e}_i^T A \underline{e}_j = \lambda_i \underline{e}_i^T \underline{e}_j$

$$\therefore (\lambda_j - \lambda_i) \underline{e}_i^T \underline{e}_j = 0$$

Because $\lambda_i \neq \lambda_j$ (distinct evs), thus $\underline{e}_i^T \underline{e}_j = 0$.

c) With $P = (\underline{e}_1 \, \underline{e}_2 \cdots \underline{e}_n)$

$$P^T P = \begin{pmatrix} \underline{e}_1^T \\ \underline{e}_2^T \\ \vdots \\ \underline{e}_n^T \end{pmatrix} (\underline{e}_1 \, \underline{e}_2 \cdots \underline{e}_n) = \begin{pmatrix} \underline{e}_1^T \underline{e}_1 & \underline{e}_1^T e_2 & \cdots & \underline{e}_1^T \underline{e}_n \\ \underline{e}_2^T \underline{e}_1 & \underline{e}_2^T \underline{e}_2 & & \vdots \\ \vdots & & & \underline{e}_n^T \underline{e}_n \end{pmatrix}$$ 7

$$= \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix} = I$$

Setter's initials: J.D.G. | Checker's initials: AOG | Page number

EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course: Paper 4

Question: EE(4)2

Marks & seen/unseen: Unseen but seen examples

Parts

$$Q = 4x_1^2 - 4x_1x_2 + x_2^2 + 6x_3^2 = \underline{x}^T A \underline{x}$$

$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \qquad \lambda_3 = 6$$

$$\lambda^2 - 5\lambda + 0 = 0$$

$$\lambda_1 = 0, \ \lambda_2 = 5.$$ 2

$\lambda_1 = 0$ $\begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0$ $q = 2p$, $r = 0$

$\underline{a}_1 = (1, 2, 0)^T/\sqrt{5}$ 4

$\lambda_2 = 5$ $\begin{pmatrix} -1 & -2 & 0 \\ -2 & -4 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0$ $p = -2q$ $r = 0$

$\underline{a}_2 = (-2, 1, 0)^T/\sqrt{5}$ 4

$\lambda_3 = 6$ $\underline{a}_3 = (0, 0, 1)^T$ 2

Note: $\underline{x}_1^T \underline{x}_2 = \underline{x}_1^T \underline{x}_3 = \underline{x}_2^T \underline{x}_3 = 0$ Check ✓.

Define $P = (\underline{a}_1 \underline{a}_2 \underline{a}_3) = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

& $\underline{x} = P\underline{y}$ so 4

$$Q = \underline{x}^T A \underline{x} = \underline{y}^T (P^T A P) \underline{y}.$$

However $P^T A P = \Lambda$ (bookwork) ← Bookwork steps

$= \text{diag}(0, 5, 6)$ ← ↙

Thus $Q = \underline{y}^T \begin{pmatrix} 0 & & \\ & 5 & \\ & & 6 \end{pmatrix} = 5y_2^2 + 6y_3^2$ 4

$\underline{y} = P^{-1}\underline{x} = P^T\underline{x} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$y_1 = \frac{1}{\sqrt{5}}(x_1 + 2x_2)$

$y_2 = \frac{1}{\sqrt{5}}(-2x_1 + x_2)$

$y_3 = x_3$

Setter's initials: JDG. Checker's initials: AOG Page number

GH/RC

Solutions

Note: U means UNSEEN

3 (i) The required probabilities are

$$(a)\quad P(X_1+X_2=3) = P\{(X_1=0\cap X_2=3)\cup(X_1=2\cap X_2=1)\}$$
$$= P(\{X_1=0\cap X_2=3\}) + P(\{X_1=2\cap X_2=1\})$$
$$= \frac{1}{2}\times\frac{3}{4}+\frac{1}{3}\times\frac{1}{4}=\frac{11}{24}$$

2 U

$$(b)\quad P(3X_1/X_2<2) = P\{(X_1=0)\cup(X_1=1\cap X_2=3)\}$$
$$= P(\{X_1=0\}) + P(\{X_1=1\})P(\{X_2=3\})$$
$$= \frac{1}{2}+\frac{1}{3}\times\frac{3}{4}=\frac{3}{4}$$

2 U

(ii) Define the random variable $X = \{$ number of completed jobs $\}$ and notice that X follows a binomial distribution with success probability $p = 1-q = 0.95$. The required probabilities are:

(a) $P(X=10) = (0.95)^{10} = 0.598$

3 U

(b) $P(X=9) = 10(0.95)^9 0.05 = 0.315$

3 U

(c) $1-P(X=10) = 1-0.598 = 0.402$

3 U

(iii) Define the random variable $X = \{$ number of breakdowns in 5 years $\}$ and notice that X follows a Poisson(λt) distribution, where $\lambda t = 0.5$. The required probability is

$$(a)\quad P(X=1) = e^{(-0.5)}0.5 = 0.303$$

3 U

Now define the random variable $Y = \{$ number of breakdowns in 1 year $\}$ and notice that Y follows a Poisson(λt) distribution, where $\lambda t = 0.1$. The required probability is

$$(b)\quad P(Y=0)+P(Y=1)+P(Y=2) = e^{-0.1}+0.1e^{-0.1}+\frac{0.1^2e^{-0.1}}{2!}=0.999$$

4 U

4. (i) (a) The hazard function is the conditional density that a t-unit-old system will fail in the imminent future given that has survived so far. It is defined as

$$h(t) = \frac{f(t)}{1-F(t)}$$

2 U

(b) For an exponential distribution we have $F(t) = 1-e^{-\lambda t}, t\geq 0$ so that $1-F(t) = e^{-\lambda t}$. Hence with $s,t>0$,

$$P(T>s+t\mid T>t) = \frac{P(T>s+t)}{P(T>t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = P(T>s)$$

3 U

(c) The hazard function in this case is

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

This is sensible since components with exponential lifetime are memoryless, and hence the probability of failure in $(t, t + \delta t)$ is independent of t

3 U

(ii) (a) Since T_A and T_B are independent, $f_{T_A,T_B}(t_A, t_B) = f_{T_A}(t_A) f_{T_B}(t_B)$

4 U

$$\begin{aligned}(b) \quad \mathrm{P}(T_A < T_B) &= \int_0^\infty dt_B \int_0^{t_B} dt_A f_{T_A,T_B}(t_A, t_B) \\ &= \int_0^\infty dt_B \int_0^{t_B} dt_A f_{T_A}(t_A) f_{T_B}(t_B) \\ &= \int_0^\infty dt_B f_{T_B}(t_B) \int_0^{t_B} dt_A f_{T_A}(t_A) \\ &= \int_0^\infty dt_B f_{T_B}(t_B) F_{T_A}(t_B) \\ &= \int_0^\infty F_{T_A}(t) f_{T_B}(t) dt\end{aligned}$$

4 U

(c) From the assumptions made, we have that $F_{T_A}(t) = 1 - e^{-\lambda_A t}$ and

$$\mathrm{P}(T_A < T_B) = \int_0^\infty [1 - e^{-\lambda_A t}] \lambda_B e^{-\lambda_B t} dt = \int_0^\infty \lambda_B e^{-\lambda_B t} dt - \lambda_B \int_0^\infty e^{-(\lambda_A + \lambda_B)t} dt$$

but $(\lambda_A + \lambda_B) \int_0^\infty e^{-(\lambda_A+\lambda_B)t} dt = 1$ because this is the integral of an exponential probability density function with parameter $\lambda_A + \lambda_B > 0$. Hence

$$\mathrm{P}(T_A < T_B) = 1 - \frac{\lambda_B}{\lambda_A + \lambda_B} = \frac{\lambda_A}{\lambda_A + \lambda_B}$$

By using symmetry,

$$\frac{\mathrm{P}(T_A < T_B)}{\mathrm{P}(T_B < T_A)} = \frac{\lambda_A}{\lambda_A + \lambda_B} \times \frac{\lambda_A + \lambda_B}{\lambda_B} = \frac{\lambda_A}{\lambda_B}$$

This is the ratio of decay parameters.

4 U (20)

5. (i) Since Y_1 is the proportion of impurities in the sample and Y_2 is the proportion of type I impurities among the sample impurities, it follows that $Y_1 Y_2$ is the proportion of type I

impurities in the entire sample. Thus we want to find

$$E(Y_1Y_2) = \int_0^1 \int_0^1 2y_1y_2(1-y_1)dy_2dy_1 = 2\int_0^1 y_1(1-y_1)\frac{1}{2}dy_1$$
$$= \int (y_1 - y_1^2)dy_1 = \left[\frac{y_1^2}{2} - \frac{y_1^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

10 U

(ii) The confidence interval for the difference of two proportions is given by

$$(\hat{p_1} - \hat{p_2}) \pm z_{\alpha/2}\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

In our case, we have $p_1 = 0.24, p_2 = 0.20, n_1 = 50, n_2 = 60$ and $z_{.01} = 2.33$. Substituting $\hat{p_1}$ and $\hat{p_2}$ for p_1 and p_2 in the computation of the standard deviation, we obtain

$$(0.24 - 0.20) \pm 2.33\sqrt{\frac{(0.24)(0.76)}{50} + \frac{(0.20)(0.80)}{60}} = 0.04 \pm 0.185 = (-0.145, 0.225)$$

10 U

6. (i) (a) They are both unbiased estimators of λ because

$$E(\Lambda_1) = \frac{1}{n}\sum_i E(X_i) = \frac{1}{n}(n\lambda) = \lambda \quad \text{and} \quad E(\Lambda_2) = \frac{1}{2}\{E(X_1) + E(X_2)\} = \frac{1}{2}(2\lambda) = \lambda$$

3 U

(b) The corresponding variances are

$$\text{Var}(\Lambda_1) = \frac{1}{n^2}\sum_i \text{Var}(X_i) = \frac{1}{n^2}(n\lambda) = \frac{\lambda}{n}$$

and

$$\text{Var}(\Lambda_2) = \frac{1}{4}(2\lambda) = \frac{\lambda}{2}$$

Thus if $n > 2$, Λ_1 is a more efficient estimator of λ than Λ_2 because $\frac{\lambda}{n} < \frac{\lambda}{2}$

5 U

(ii) (a)

$$E(y_t) = 0 \qquad \text{Var}(y_t) = \left(1 + \frac{1}{4} + \frac{1}{16}\right)\sigma_e^2 = \frac{21}{16}\sigma_e^2$$

3 U

(b) $$\mathrm{Cov}(y_t, y_{t-s}) = \mathrm{Cov}\left(e_t + \frac{1}{2}e_{t-1} + \frac{1}{4}e_{t-2}, e_{t-s} + \frac{1}{2}e_{t-s-1} + \frac{1}{4}e_{t-s-2}\right)$$
$$= \begin{cases} \left(\frac{1}{2} + \frac{1}{8}\right)\sigma_e^2 = \frac{5}{8}\sigma_e^2 & \text{for} s = 1 \\ \frac{1}{4}\sigma_e^2 & \text{for} s = 2 \\ 0 & \text{for} s > 2 \end{cases}$$

6 U

(c) The process is stationary because mean, variance and covariance are independent of t

3 U