EE - stream (E2.9)

UNIVERSITY OF LONDON

[II(4)E 2002]

B.ENG. AND M.ENG. EXAMINATIONS 2002

For Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examination for the Associateship.

PART II : MATHEMATICS 4 (ELECTRICAL ENGINEERING)

Thursday 30th May 2002 2.00 - 4.00 pm

Answer FOUR questions.

[Before starting, please make sure that the paper is complete; there should be 4 pages, with a total of 6 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. Find the eigenvalues and normalised eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 1 & \sqrt{2} \\ 0 & \sqrt{2} & 1 \end{pmatrix}.$$

By writing the quadratic form

$$Q = x_1^2 + 2\sqrt{2}\,x_1x_2 + x_2^2 + 2\sqrt{2}\,x_2x_3 + x_3^2$$

as

$$Q = \mathbf{x}^T A \mathbf{x},$$

where $\mathbf{x} = (x_1, x_2, x_3)^T$, show that Q can be written in the diagonal form

$$Q = 3y_1^2 + y_2^2 - y_3^2 ,$$

by finding a matrix P which satisfies $\mathbf{x} = P\mathbf{y}$ where $\mathbf{y} = (y_1, y_2, y_3)^T$.

Find y_1, y_2 and y_3 in terms of x_1, x_2 and x_3 from the matrix P.

2. Consider a real $n \times n$ symmetric matrix A with distinct eigenvalues λ_i and corresponding normalised eigenvectors $\boldsymbol{a}_i$ for $i = 1, \ldots n$.

(i) Show that all the λ_i are real.

(ii) Show that the eigenvectors $\boldsymbol{a}_i$ obey the orthogonality relation

$$\boldsymbol{a}_i^T \boldsymbol{a}_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

(iii) Show that the $n \times n$ matrix $P = \{\boldsymbol{a}_1\, \boldsymbol{a}_2\, \ldots\, \boldsymbol{a}_n\}$ satisfies the relation

$$P^T P = I,$$

where I is the $n \times n$ unit matrix.

PLEASE TURN OVER

3. (i) Draw a Venn diagram to illustrate the configuration of events when $B \subset \{A_1 \cup A_2 \cup A_3\}$, the As being mutually exclusive. Write down an expression for $\mathrm{pr}(B)$ in terms of the quantities $\mathrm{pr}(B \mid A_j)$ and $\mathrm{pr}(A_j)$ ($j = 1, 2, 3$).

   (ii) Events C and D have conditional probabilities $\mathrm{pr}(C \mid D) = \frac{1}{2}$ and $\mathrm{pr}(D \mid C) = \frac{1}{3}$. Calculate $\mathrm{pr}(C)/\mathrm{pr}(D)$ and $\mathrm{pr}(C \cup D)\ /\mathrm{pr}(C)$.

   (iii) Certain electrical components can be classified into three quality-bands, high, medium and low. Supplier A provides 80% of components to the market, in proportions 0.75 (high), 0.20 (medium) and 0.05 (low). Supplier B provides 20% in proportions 0.65, 0.30 and 0.05. An otherwise unidentified component is tested and found to be of medium quality. What is the probability that it was supplied by Supplier A?

4. (i) The discrete random variables X_1 and X_2 are independent and, for $j = 1, 2$, X_j has probability function $p_j(x_j)$ for $x_j = 0, 1, 2, \ldots$. Derive the convolution formula $\mathrm{pr}(X_1 + X_2 = r) = \sum_{s=0}^{r} p_1(s)p_2(r-s)$.

   (ii) The discrete random variable X_1 takes values 0, 1 and 2 with respective probabilities 1/2, 1/3 and 1/6; X_2 takes values 1 and 3 with probabilities 1/4 and 3/4; X_1 and X_2 are independent.
   Compute $\mathrm{pr}(X_1 + X_2 = 3)$ and $\mathrm{pr}(3X_1/X_2 < 2)$.

   (iii) The peak power over one day required by a certain machine is a continuous random variable with density function $f(y) = 6ye^{-3y^2}$ on $(0, \infty)$. Calculate the distribution function $F(y)$ of the peak power and hence find its median. Compute the probability that the peak power exceeds 1.0. Assuming that peak power requirements on different days are independent, what is the probability that the peak power will remain below 1.0 over seven days?

5. (i) If T_1 and T_2 are independent failure-time variates, with respective hazard functions h_1 and h_2, show that $\min(T_1, T_2)$ has hazard function $h_1(t)+h_2(t)$. Calculate the survivor function of $\min(T_1, T_2)$ when $h_1(t) = \alpha$ and $h_2(t) = 2\beta t$.

(ii) The degradation $Y(t)$ of a certain electrical system at time t is described by the curve $Y(t) = Z(1 - e^{-ct})$, where c is a positive constant and Z is a random variable with density function $f_Z(z) = 6(1 + 2z)^{-4}$ on $(0, \infty)$. The system is classified as failed when $Y(t)$ reaches the critical level y_0.
Calculate $\mathrm{pr}(T \le t)$, T being the time to failure of the system. Allow $t \to \infty$ in your answer and identify the event of which this is the probability. Also, relate this result to the degradation curve and the distribution of Z.

6. (i) The random variables X and Y have joint density function

$$f(x, y) = 24(1 + 2x + 2y)^{-4} \text{ on } \{x > 0, y > 0\}.$$

Calculate the marginal density of X and its distribution function. Verify that $E(X) = \frac{1}{2}$ and, noting that $f(x, y)$ is symmetric in x and y, find $E(X + Y)$ and $\mathrm{pr}(Y > 1)$.

(ii) A system has 'strength' X and 'stress' Y is placed upon it: X and Y are independent and the system 'fails' if $Y > X$. Show that the system failure probability is given by

$$\mathrm{pr}(Y > X) = \int_0^\infty f_Y(y) F_X(y)\, dy,$$

where $f_Y(y)$ is the density function of Y and $F_X(x)$ is the distribution function of X.

(iii) A circuit-breaker will trip out if the current exceeds X, where, due to imperfect manufacturing quality control, X varies among the supplied units with distribution function $F_X(x) = 1 - e^{-\lambda(x-a)}$ on (a, ∞), a being a positive constant. Suppose that the current Y has distribution function $F_Y(y) = 1 - e^{-\mu y}$ on $(0, \infty)$, where μ is a positive constant, and is independent of X. Show that the probability that the unit will trip out is given by

$$\int_a^\infty \mu e^{-\mu y} \left\{1 - e^{-\lambda(y-a)}\right\} dy$$

and evaluate this integral.

END OF PAPER

2nd yr data-sheet

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

Data sheet

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\cos b + \cos a\sin b\,;$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^ng + \binom{n}{1}Df\,D^{n-1}g + \ldots + \binom{n}{r}D^rf\,D^{n-r}g + \ldots + D^nf\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^nf^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

September 2000

$$A = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 1 & \sqrt{2} \\ 0 & \sqrt{2} & 1 \end{pmatrix} \quad \therefore \quad \begin{vmatrix} 1-\lambda & \sqrt{2} & 0 \\ \sqrt{2} & 1-\lambda & \sqrt{2} \\ 0 & \sqrt{2} & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(\lambda-1)^2 - 2\right] - \sqrt{2}\left[\sqrt{2}(1-\lambda)\right] = 0 \Rightarrow (\lambda-1)(\lambda+1)(\lambda-3) = 0$$

$\lambda_1 = 3, \quad \lambda_2 = 1, \quad \lambda_3 = -1$ 3

$\lambda_1 = 3: \quad \underline{x}_1 = c_1 (1 \ \ \sqrt{2} \ \ 1)^T \qquad c_1 = \frac{1}{2}$

$\lambda_2 = 1: \quad \underline{x}_2 = c_2 (1 \ \ 0 \ \ -1)^T \qquad c_2 = 1/\sqrt{2}$ 6

$\lambda_3 = -1: \quad \underline{x}_3 = c_3 (1 \ \ -\sqrt{2} \ \ 1)^T \qquad c_3 = 1/2$

For $\underline{x} = P\underline{y}$ so $\underline{x}^T = \underline{y}^T P^T$

$\therefore \quad Q = \underline{x}^T A \underline{x} = \underline{y}^T (P^T A P) \underline{y}$ 2

Now, rewrite the e-equations $A\underline{x}_i = \lambda_i \underline{x}_i$ as

$$AP = P\Lambda \qquad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$$

where $P = \{\underline{x}_1 \ \underline{x}_2 \ \underline{x}_3 \ \cdots \ \underline{x}_n\}$. P has the property. 2

or, directly.

$$P^T P = \begin{pmatrix} \underline{x}_1^T \\ \underline{x}_2^T \\ \vdots \\ \underline{x}_n^T \end{pmatrix} (\underline{x}_1 \ \underline{x}_2 \ \cdots \ \underline{x}_n) = \{\underline{x}_i^T \underline{x}_j\} = I$$

$$\therefore \quad P^T A P = P^T P \Lambda = \Lambda = \mathrm{diag}(3, 1, -1)$$

$$\therefore \quad \underline{Q = \underline{y}^T \Lambda \underline{y} = 3y_1^2 + y_2^2 - y_3^2}$$ 2

$\underline{x} = P\underline{y} \Rightarrow \underline{y} = P^{-1}\underline{x} = P^T \underline{x}$ $\qquad P = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$ 2

$y_1 = \frac{1}{2}x_1 + \frac{1}{\sqrt{2}}x_2 + \frac{1}{2}x_3$

$y_2 = \frac{1}{\sqrt{2}}(x_1 - x_3)$ $\qquad P^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$

$y_3 = \frac{1}{2}x_1 - \frac{1}{\sqrt{2}}x_2 + \frac{1}{2}x_3$ 3

Check: $P^T P = I$

Setter: J.D. GIBBON Setter's signature: J.D. Gibbon

Checker: HERBERT Checker's signature: D.R. Herbert

1) $A\underline{a}_i = \lambda_i \underline{a}_i$ Assume $\lambda_i, \underline{a}_i$ are complex.

L.H. multiply by $\underline{a}_i^{*T}$: $\underline{a}_i^{*T} A \underline{a}_i = \lambda_i \underline{a}_i^{*T} \underline{a}_i$;

Complex conjugate, transpose & R.H. multiply by $\underline{a}_i$: $\underline{a}_i^{*T} A_i^{*T} \underline{a}_i = \lambda_i^* \underline{a}_i^{*T} \underline{a}_i$ — 6

If $A^{*T} = A$ then $\lambda_i = \lambda_i^* \Rightarrow \lambda_i$ real.

2) Now take two sets of evals/evecs: $(\lambda_i, \underline{a}_i)$ & $(\lambda_j, \underline{a}_j)$

$A\underline{a}_i = \lambda_i \underline{a}_i$ $\qquad$ $A\underline{a}_j = \lambda_j \underline{a}_j$

LH. multiply by $\underline{a}_j^T$ $\qquad$ Transpose & R.H. multiply by $\underline{a}_i$

$\underline{a}_j^T A \underline{a}_i = \lambda_i \underline{a}_j^T \underline{a}_i$ $\qquad$ $\underline{a}_j^T A^T \underline{a}_i = \lambda_j \underline{a}_j^T \underline{a}_i$ — 8

If $A^T = A$ then subtract; $(\lambda_i - \lambda_j)\underline{a}_j^T \underline{a}_i = 0$.

Since $\lambda_i \neq \lambda_j \Rightarrow$ $\underline{a}_j^T \underline{a}_i = 0 \quad i \neq j$

$\underline{a}_i^T \underline{a}_i = 1$ (given) $i = j$.

3) $$P = \begin{pmatrix} \underline{a}_1^T \\ \underline{a}_2^T \\ \vdots \\ \underline{a}_n^T \end{pmatrix} (\underline{a}_1 \ \underline{a}_2 \ \cdots \ \underline{a}_n) = \{\underline{a}_i^T \underline{a}_j\} = I$$ — 6

<u>Bookwork.</u>

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

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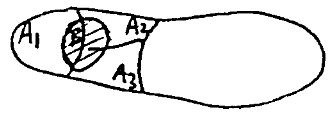
PAPER EE2 (4)

QUESTION

SOLUTION 3

(i) Venn diagram as seen. 

2

$\mathrm{pr}(B) = \sum_{j=1}^{3} \mathrm{pr}(B \mid A_j)\ \mathrm{pr}(A_j)$ 2

(ii) $\mathrm{pr}(C \mid D) = \mathrm{pr}(C \cap D)/\mathrm{pr}(D)$ and $\mathrm{pr}(D \mid C) = \mathrm{pr}(D \cap C)/\mathrm{pr}(C)$

so $\mathrm{pr}(C)/\mathrm{pr}(D) = \mathrm{pr}(C \mid D)/\mathrm{pr}(D \mid C) = 3/2$ 4

and $\mathrm{pr}(C \cup D)/\mathrm{pr}(C) = \{\mathrm{pr}(C) + \mathrm{pr}(D) - \mathrm{pr}(C \cap D)\}/\mathrm{pr}(C)$

$= 1+(2/3)-(1/3) = 4/3.$ 4

(iii) prob $= \mathrm{pr}(A \mid medium) = \mathrm{pr}(medium \mid A)\ \mathrm{pr}(A)/\mathrm{pr}(medium)$ 3

and $\mathrm{pr}(medium) = \mathrm{pr}(medium \mid A)\ \mathrm{pr}(A) + \mathrm{pr}(medium \mid B)\ \mathrm{pr}(B)$

$= (0.20 \times 0.8) + (0.30 \times 0.2) = 0.22$ 3

so prob $= (0.20 \times 0.8)/0.22 = 8/11 = 0.727272.$ 2

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Setter : MJ CROWDER   Setter's signature : MJ Crowder

Checker : AT WALDEN   Checker's signature : ATWalden

PAPER EE2 (4)

QUESTION

SOLUTION 4

(i) $\mathrm{pr}(X_1 + X_2 = r) = \sum_{s=0}^{r} \mathrm{pr}(X_1 = s \cap X_2 = r - s)$

$= \sum_{s=0}^{r} \mathrm{pr}(X_1 = s)\mathrm{pr}(X_2 = r - s) = \text{result}$ 3

(ii) $\mathrm{pr}(X_1 + X_2 = 3) = \mathrm{pr}(X_1 = 0 \cap X_2 = 3) + \mathrm{pr}(X_1 = 2 \cap X_2 = 1)$

$= (1/2 \times 3/4) + (1/6 \times 1/4) = 5/12 = \cancel{0.454545}$ 0.416667 3

$\mathrm{pr}(3X_1/X_2 < 2) = \mathrm{pr}(X_1 = 0) + \mathrm{pr}(X_1 = 1 \cap X_2 = 3)$

$= 1/2 + (1/3 \times 3/4) = 3/4$ 3

(iii) $F(y) = \int_0^y f(u)du = 1 - e^{-3y^2}$ 3

$F(median) = 1/2 \quad \Rightarrow \quad median = \sqrt{-(1/3)\log(1/2)} = 0.4807$ 3

$\mathrm{pr}(exceeds\ 1.0) = 1 - F(1.0) = 0.0498$ 2

$\mathrm{pr}(below\ 1.0\ on\ 7\ days) = (1 - 0.0498)^7 = 0.6994$ 3

20

Setter : MJ CROWDER Setter's signature : MJ Crowder

Checker : AT WALDEN Checker's signature : AWalden

PAPER EE2(4)

QUESTION

SOLUTION 5

EE-2002

Ans 3.

(i) $\mathrm{pr}\{\min(T_1, T_2) > t\} = \mathrm{pr}(T_1 > t) \times \mathrm{pr}(T_2 > t)$

$= \exp\{-\int_0^t h_1(s)ds\} \times \exp\{-\int_0^t h_2(s)ds\}$

$= \exp\{-\int_0^t [h_1(s) + h_2(s)]ds\} \Rightarrow$ result 4

$\mathrm{pr}\{\min(T_1, T_2) > t\} = \exp\{-\int_0^t (\alpha + 2\beta s)ds\} = \exp(-\alpha t - \beta t^2)$ 3

(ii) $\mathrm{pr}(T \leq t) = \mathrm{pr}\{Y(t) \geq y_0\} = \mathrm{pr}\{Z \geq y_0/(1 - \mathrm{e}^{-ct})\}$ 3

$= \int_{y_0/(1-\mathrm{e}^{-ct})}^{\infty} f_Z(z)dz = [-(1+2z)^{-3}]_{y_0/(1-\mathrm{e}^{-ct})}^{\infty}$

$= \{1 + 2y_0/(1 - \mathrm{e}^{-ct})\}^{-3}$ 4

$t \to \infty \Rightarrow \mathrm{pr}(T\ finite) = (1 + 2y_0)^{-3}$ 2

identify: $(1 + 2y_0)^{-3}$ is $\mathrm{pr}(Z > y_0)$ 2

event $\{Z > y_0\}$ is that the curve, which rises asymptotically to Z, crosses level y_0 2

20

Setter : M.J. CROWDER — Setter's signature : M.J. Crowder

Checker : A.T. WALDEN — Checker's signature : A T Walden

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

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PAPER EE2(4)

QUESTION

SOLUTION 6

(i) marginal density $f_X(x) = [-4(1+2x+2y)^{-3}]_{y=0}^{\infty}$

$= 4(1+2x)^{-3}$ on $(0, \infty)$. 2

distn fn $F_X(x) = [-(1+2x)^{-2}]_0^x = 1-(1+2x)^{-2}$ 2

$\mathrm{E}(X) = \int_0^\infty 4x(1+2x)^{-3}dx =$ (by parts)

$= [-x(1+2x)^{-2}]_0^\infty + \int_0^\infty (1+2x)^{-2}dx = 0 + [-\frac{1}{2}(1+2x)^{-1}]_0^\infty = \frac{1}{2}$ 3

$\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(X)$ (since distn symmetric in x and y) $=1$ 2

$\mathrm{pr}(Y>1) = \mathrm{pr}(X>1) = 1 - F_X(1) = 1/9$ 2

(ii) $\mathrm{pr}(Y>X) = \int_{y>x} f(x,y)dxdy = \int_{y>x} f_X(x)f_Y(y)dxdy$

$= \int_0^\infty dy \int_0^y dx\{f_X(x)f_Y(y)\} = \int_0^\infty dy\{f_Y(y)F_X(y)\}$ 4

(iii) $\mathrm{pr}(trip - out) = \mathrm{pr}(Y>X) = \int_0^\infty f_Y(y)F_X(y)dy =$ result given 2

$= [-\mathrm{e}^{-\mu y} + \frac{\mu}{\mu+\lambda}\mathrm{e}^{-(\mu+\lambda)y+\lambda a}]_a^\infty = \mathrm{e}^{-\mu a} - \frac{\mu}{\mu+\lambda}\mathrm{e}^{-\mu a} = \frac{\lambda}{\lambda+\mu}\mathrm{e}^{-\mu a}.$ 3

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Setter : MJ CROWDER

Setter's signature : MJ Crowder

Checker : AT WALDEN

Checker's signature : ATWalden