UNIVERSITY OF LONDON [II(4)E 2001]

B.ENG. AND M.ENG. EXAMINATIONS 2001

For Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examination for the Associateship.

PART II : MATHEMATICS 4 (ELECTRICAL ENGINEERING)

Thursday 7th June 2001 2.00 - 4.00 pm

Answer FOUR questions.

[*Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 6 questions. Ask the invigilator for a replacement if your copy is faulty.*]

[II(4)E 2001]

1. Find the eigenvalues and normalized eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} .$$

Using these, or otherwise, show that the matrix

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

diagonalises A such that

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} .$$

2. Show that the quadratic form

$$Q = 4x_1^2 + 4x_1x_2 + x_2^2 + 4x_3^2$$

can be written as

$$Q = \mathbf{x}^T A \mathbf{x} ,$$

where $\mathbf{x} = (x_1, x_2, x_3)^T$ and A is a real symmetric matrix, which is to be found. Hence, show that Q can be re-expressed in the diagonal form

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 ,$$

where the λ_i are to be determined, by finding a matrix P that satisfies $\mathbf{x} = P\mathbf{y}$, where $\mathbf{y} = (y_1, y_2, y_3)^T$. Find y_1, y_2 and y_3 in terms of x_1, x_2 and x_3 from the matrix P.

PLEASE TURN OVER

[II(4)E 2001]

3. Use the definition of conditional probability to show that $P(A|B)P(B) = P(B|A)P(A)$.

A packet-switching network is, in any day, subject to either standard or heavy use, with probabilities 0.85 and 0.15, respectively. At the end of a day it is in one of three mutually exclusive states: operational, out of order due to a software failure or out of order due to a hardware failure. Given standard use, the probabilities of software and hardware failures are 0.05 and 0.015, respectively. Under heavy use both these probabilities are tripled.

(i) Find the probability that the network is not operational at the end of a day given

(a) standard use,

(b) heavy use,

during the day.

(ii) Show that the probability that the network is operational at the end of a day is 0.9155.

(iii) If the network is not operational at the end of a day, find the probability that it was subjected to heavy use during the day.

(iv) Given that the network is not operational at the end of the day, find the probability that it experienced software failure during the day.

[II(4)E 2001]

4. Explain the terms *prior distribution* and *posterior distribution*, in relation to the Bayesian analysis of data, and show how they are related.

   Lifetimes, T, for certain components have a probability density function

   $$f(t) \;=\; \beta^2 t e^{-\beta t}, \quad 0 \le t < \infty, \, \beta > 0\,,$$

   and β has a prior probability density function

   $$\Pi\,(\beta) \;=\; \lambda e^{-\lambda\beta}, \quad 0 < \beta < \infty\,,$$

   where λ is a known constant.

   (i) Write down the likelihood function (as a function of β) of a sample $t_1, \ldots, t_n$ of lifetimes.

   (ii) Show that the posterior probability density function of β has the form

   $$f(\beta | t_1, \ldots, t_n) \propto \beta^{2n} \exp\left\{-\beta\left(\lambda \;+\; \sum_{i=1}^{n} t_i\right)\right\}\,, \quad 0 < \beta < \infty\,,$$

   and show that the constant of proportionality, C, say, has the form

   $$C \;=\; \frac{(\lambda \;+\; \sum_{i=1}^{n} t_i)^{2n+1}}{(2n)!}\;.$$

   (ii) Find the posterior mean corresponding to the posterior probability density function $f\,(\beta | t_1, \ldots, t_n)$.

You may use the fact that for any positive integer n, $\int_0^\infty x^n e^{-x} dx = n!$

PLEASE TURN OVER

5. Let the positive random variable T (in hours) represent the lifetime of an electrical component. Carefully define the hazard rate $z(t)$ of the component and derive an expression for it in terms of the density and distribution functions of T.

   (i) Show that for any hazard rate function $z(t)$, the reliability $R(t)$ is given by

   $$R(t) \;=\; \exp\left[-\int_0^t z(x)dx\right] .$$

   (ii) If $z(t)$ has the Weibull form $z(t) = \alpha\beta t^{\alpha-1}$, where α, β are positive constants:

   (a) Find the reliability, $R(t)$.

   (b) Show that the probability density function $f(t)$ of the Weibull random variable T takes the form

   $$f(t) \;=\; \alpha\beta t^{\alpha-1}e^{-\beta t^{\alpha}}, \quad t \geq 0 .$$

   (c) Given that for $x > 0$,

   $$\int_0^\infty u^{x-1}e^{-u}du \;=\; \Gamma(x) ,$$

   where $\Gamma(x)$ is the Gamma function, show that the mean, $E\{T\}$, of T is given by

   $$E\{T\} \;=\; \beta^{-\frac{1}{\alpha}}\Gamma\left(1 \,+\, \frac{1}{\alpha}\right) .$$

   (d) For which value of the Weibull distribution parameter α is the exponential distribution obtained? Given that $\Gamma(n+1) \;=\; n!$ for non-negative integers n, show that the mean obtained in (c) is consistent with that of an exponential random variable.

6. Given two random variables X and Y with joint probability density function $f_{X,Y}(x,\, y)$, the minimum mean square error estimate of the unobserved value $Y = y$ in terms of the observed value $X = x$ is given by $E\{Y \mid X = x\}$.

   (i) Carefully explain the meaning of this statement.

   The random variables X and Y which represent the amplitudes of two signals, have joint probability density function

$$f_{X,Y}(x,\, y) = \begin{cases} C(x \; + \; 2y)\,, & 0 \le x \le 2;\, 0 \le y \le 1\,, \\ \\ 0\,, & \text{otherwise}\,, \end{cases}$$

   where C is a constant.

   (ii) Find the value of C required for such a joint probability density function.

   (iii) Find $f_{Y|X=x}(y|x)$, the conditional probability density function of Y, given $X = x$, and show that it integrates to unity.

   (iv) Hence derive the minimum mean square error estimate of $Y = y$ in terms of $X = x$.

END OF PAPER

MATHS 4 - 2001

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

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PAPER 4EE

QUESTION

SOLUTION 1

$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ $\lambda_3 = 4$ & $\lambda_{1,2}$ .. satisfy

$(\lambda-2)^2 - 1 = 0$ $\lambda_2 = 3,\ \lambda_1 = 1$

$\lambda_3 = 4;$ $\left.\begin{array}{l} 2a = b \\ 2b = a \end{array}\right\} \Rightarrow a = b$ & c arb. $\therefore$ $\underline{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ 6

$\lambda_2 = 3$ $\begin{array}{l} a = b \\ c = 0 \end{array}$ $\underline{a}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_1 = 1$ $\begin{array}{l} b = -a \\ c = 0 \end{array}$ $\underline{a}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

A is symmetric

$\therefore$ $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} = (\underline{a}_1, \underline{a}_2, \underline{a}_3)$ $\underline{a}_i^T \underline{a}_j = \delta_{ij}$ 6

Either

$\therefore$ $P^T P = \{\underline{a}_i^T \underline{a}_j\} = \{\delta_{ij}\} = I \Rightarrow P^T = P^{-1}$

Now $A\underline{a}_i = \lambda_i \underline{a}_i \Rightarrow AP = P\Lambda \Rightarrow P^{-1}AP = \Lambda$

$\therefore$ $P^T A P = \Lambda$ $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}$ 3

OR (directly)

$AP = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 3 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 4\sqrt{2} \end{pmatrix}$

$P^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$

$\therefore$ $P^T A P = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 4\sqrt{2} \end{pmatrix}$ 5

$= \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Setter : J.D. GIBBON Setter's signature : J.D. Gibbon

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

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PAPER 4

QUESTION

SOLUTION 2

$$Q = 4x_1^2 + 4x_1x_2 + x_2^2 + 4x_3^2 = \underline{x}^T \begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \underline{x}$$

Evs of A: $\lambda = 4$ & $(\lambda-4)(\lambda-1) - 4 = 0$ so $\lambda = 0, 5$.

$\lambda_1 = 0 \quad \lambda_2 = 4 \quad \lambda_3 = 5$

$\underline{a}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \quad \underline{a}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \underline{a}_3 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad \underline{a}_i^T \underline{a}_j = \delta_{ij}$ as $A^T = A$ 8

Write $P = \{\underline{a}_1, \underline{a}_2, \underline{a}_3\}$

then $P^T P = \{\underline{a}_i^T \underline{a}_j\} = \{\delta_{ij}\} = I$

$\therefore \quad P^T = P^{-1}$

Writing $\underline{x} = P\underline{y} \Rightarrow \underline{y} = P^T \underline{x}$ 4

and

$$Q = \underline{x}^T A \underline{x} = (P\underline{y})^T A (P\underline{y}) = \underline{y}^T (P^T A P) \underline{y}$$

Moreover $A\underline{x}_i = \lambda_i \underline{x}_i \Rightarrow AP = P\Lambda \qquad \Lambda = \begin{pmatrix} 0 & & \\ & 4 & \\ & & 5 \end{pmatrix}$

so $P^T A P = \Lambda$ 4

$\therefore \quad Q = \underline{y}^T \Lambda \underline{y}$

$= 4y_2^2 + 5y_3^2$.

and $\underline{y} = P^T \underline{x}$. where $P = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 0 & 2 \\ -2 & 0 & 1 \\ 0 & \sqrt{5} & 0 \end{pmatrix}$

$\therefore \quad y_1 = \frac{1}{\sqrt{5}}(x_1 - 2x_2) \qquad P^T = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & \sqrt{5} \\ 2 & 1 & 0 \end{pmatrix}$. 4

$y_2 = x_3$

$y_3 = \frac{1}{\sqrt{5}}(2x_1 + x_2)$

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PAPER: EE II (4)

QUESTION

SOLUTION: Stats 3

By definition $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$, but $P(A\cap B) = P(B\cap A)$

and $P(B|A) = \dfrac{P(B\cap A)}{P(A)}$ so that $P(A|B)P(B) = P(B|A)P(A)$. 4

Let

S = standard use ; O = operational ; R = software fail ; H = hardware fail

$P(R|S) = 0.05$ $\quad P(H|S) = 0.015$ 2

$P(R|\bar{S}) = 0.15$ $\quad P(H|\bar{S}) = 0.045$

$P(S) = 0.85$ $\quad P(\bar{S}) = 0.15$

(a) $P(\bar{O}|S) = P(R|S) + P(H|S) = 0.05 + 0.015 = \underline{0.065}$. 2

$P(\bar{O}|\bar{S}) = P(R|\bar{S}) + P(H|\bar{S}) = 0.15 + 0.045 = \underline{0.195}$ 2

(b) $P(O) = P(O|S)P(S) + P(O|\bar{S})P(\bar{S})$

$= (1-P(\bar{O}|S))P(S) + (1-P(\bar{O}|\bar{S}))P(\bar{S})$

$= (.935 \times .85) + (.805 \times .15) = \underline{.9155}$ 4

(c) $P(\bar{S}|\bar{O}) = P(\bar{O}|\bar{S})P(\bar{S})/P(\bar{O}) = \dfrac{P(\bar{O}|\bar{S})P(\bar{S})}{1-P(O)} = \dfrac{.195\times.15}{.0845}$

$= \underline{.3462}$ to 4 dp 4

(d) $P(R|\bar{O}) = \dfrac{P(R\cap\bar{O})}{P(\bar{O})} = \dfrac{P(R)}{P(\bar{O})} = \dfrac{.065}{.0845} = \underline{.7692}$ to 4 dp 2

Since $P(R) = P(R|S)P(S) + P(R|\bar{S})P(\bar{S})$

$= (.05 \times .85) + (.15 \times .15)$ 2

$= .065$

Setter : A.T. WALDEN Setter's signature : ATW.

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If β is an unknown parameter, the prior distribution of β represents our prior knowledge on β before we collect the data. The posterior represents our knowledge of β after we've collected the data. The prior is modified by the likelihood of the data, given β, to give the posterior distribution

$$f(\beta \mid \text{data}) \propto f(\text{data} \mid \beta)\,\pi(\beta)$$ 5

$f(t) = \beta^2 t e^{-\beta t}$ and $\pi(\beta) = \lambda e^{-\lambda\beta}$ so

(i) $L = \prod_{i=1}^{n} f(t_i) = \beta^{2n}\left(\prod_{i=1}^{n} t_i\right) e^{-\beta\sum_{i=1}^{n} t_i}$ 3

(ii) $\pi(\beta \mid \underline{t}) \propto \beta^{2n} e^{-\beta\sum_{i=1}^{n} t_i} \lambda e^{-\lambda\beta} \propto \beta^{2n} e^{-\beta(\lambda + \sum t_i)}$ 4

$0 < \beta < \infty.$

But $\int_0^\infty x^n e^{-x}\,dx = n!$ so

$$c\int_0^\infty \beta^{2n} e^{-\beta(\lambda+\sum t_i)}\,d\beta = 1 \Rightarrow \frac{c}{\lambda+\sum t_i}\int_0^\infty \frac{x^{2n}}{(\lambda+\sum t_i)^{2n}} e^{-x}\,dx = 1$$

$$\Rightarrow \frac{c(2n)!}{(\lambda+\sum t_i)^{2n+1}} = 1 \Rightarrow c = (\lambda+\sum t_i)^{2n+1}/(2n)!.$$ 4

(iii)

$$E\{\beta \mid \underline{t}\} = c\int_0^\infty \beta^{2n+1} e^{-\beta(\lambda+\sum t_i)}\,d\beta$$

$$= \frac{c}{(\lambda+\sum t_i)^{2n+2}}\int_0^\infty x^{2n+1} e^{-x}\,dx$$

$$= \frac{c}{(\lambda+\sum t_i)^{2n+2}}(2n+1)! = \frac{2n+1}{\lambda+\sum t_i}.$$ 4

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PAPER EE II (4)
QUESTION
SOLUTION 5 Stats

a) Suppose a component has survived for t time units and we want the probability it will not survive for an additional time Δt. Then, with T the time to failure,

$$P(T \in [t, t+\Delta t] \mid T > t) = P(T \in [t, t+\Delta t]) / P(T > t)$$
$$= f(t)\,\Delta t \,/\, [1 - F(t)]$$

The hazard rate $z(t) = f(t)/[1-F(t)]$ represents the conditional probability intensity that a t-unit-old system will fail. 5

$$\int_0^t z(x)\,dx = \int_0^t \frac{f(x)}{1-F(x)}\,dx = -\int_0^t \frac{u'(x)}{u(x)}\,dx \quad \text{where } u(x) = 1-F(x)$$
$$= -\ln[1-F(t)]$$

So $R(t) = 1 - F(t) = \exp\left[-\int_0^t z(x)\,dx\right]$. 4

b) (i) $R(t) = \exp\left[-\int_0^t \alpha\beta x^{\alpha-1}\,dx\right] = \exp\left[-\beta x^\alpha \Big|_0^t\right] = e^{-\beta t^\alpha}$. 2

(ii) $z(t) = f(t)/R(t)$ so $f(t) = z(t)R(t) = \alpha\beta t^{\alpha-1} e^{-\beta t^\alpha}$. 2

(iii) $E\{T\} = \int_0^\infty t\alpha\beta t^{\alpha-1} e^{-\beta t^\alpha}\,dt = \int_0^\infty \alpha\beta t^\alpha e^{-\beta t^\alpha}\,dt$. Put $u = \beta t^\alpha$ so $t = \left(\frac{u}{\beta}\right)^{1/\alpha}$, $du = \alpha\beta\left(\frac{u}{\beta}\right)^{1-1/\alpha} dt$ and $\therefore\ dt = \frac{1}{\alpha\beta}\left(\frac{u}{\beta}\right)^{\frac{1}{\alpha}-1} du$ and

$$E\{T\} = \int_0^\infty \alpha\beta\left(\frac{u}{\beta}\right) e^{-u} \frac{1}{\alpha\beta}\left(\frac{u}{\beta}\right)^{1/\alpha - 1} du = \beta^{-1/\alpha}\int_0^\infty u^{1/\alpha} e^{-u}\,du = \beta^{-1/\alpha}\,\Gamma(1 + 1/\alpha).$$ 4

(iv) The pdf in (ii) matches the exponential for $\alpha = 1$.
Then $E\{T\} = \beta^{-1}\Gamma(2) = \beta^{-1}1! = 1/\beta$
as for the exponential distribution. 3

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

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PAPER EE II(4)

QUESTION

SOLUTION 6

Stats

(i) The minimum mean square error estimate of $Y=y$ in terms of $X=x$ is got by finding that function $c(x)$ s.t.

$$E\{(Y-c(X))^2\} = \iint (y-c(x))^2 f_{X,Y}(x,y)\,dx\,dy = \min.$$

It turns out that $c(x) = E\{Y|X=x\}$ is the minimizing fn. 5

$$f_{X,Y}(x,y) = \begin{cases} c(x+2y), & 0 \le x \le 2;\ 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(ii) $\int_{x=0}^{2}\int_{y=0}^{1} c(x+2y)\,dx\,dy = c\int_{x=0}^{2} \{xy+y^2\}\Big|_0^1 dx = c\int_{x=0}^{2} 1+x\,dx$

$= c\int_0^2 \frac{x^2}{2} + x\,dx = 4C \Rightarrow \underline{C = 1/4}.$ 5

(iii) $f_{Y|X=x}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{x+2y}{x+1}$ $\quad 0 \le x \le 2;\ 0 \le y \le 1$ 2

since $f_X(x) = \frac{1}{4}\int_0^1 (x+2y)\,dy = xy+y^2\Big|_0^1/4 = \frac{x+1}{4}$, $0 \le x \le 2$. 1

And

$\int_0^1 f_{Y|X=x}(y|x)\,dy = \int_0^1 \frac{x+2y}{x+1}\,dy = \frac{1}{x+1}\left[xy+y^2\right]_0^1 = \frac{x+1}{x+1} = 1.$ 2

5

(iii)

$E\{Y|X=x\} = \int_0^1 y\left(\frac{x+2y}{x+1}\right)dy = \frac{1}{x+1}\int_0^1 yx+2y^2\,dy$

$= \frac{1}{x+1}\left[\frac{xy^2}{2} + \frac{2}{3}y^3\right]_0^1 = \frac{1}{x+1}\left\{\frac{x}{2}+\frac{2}{3}\right\}.$ 5

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