Master

UNIVERSITY OF LONDON

[E2.8 (Maths 3) 2007]

B.ENG. AND M.ENG. EXAMINATIONS 2007

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART II : MATHEMATICS 3 (ELECTRICAL ENGINEERING)

Wednesday 30th May 2007   2.00 - 5.00 pm

Answer EIGHT questions.

Please answer questions from Section A and Section B in separate answerbooks.

Mathematical formulae and Statistics data sheets are provided.

[*Before starting, please make sure that the paper is complete; there should be 8 pages, with a total of 12 questions. Ask the invigilator for a replacement if your copy is faulty.*]

SECTION A

[E2.8 (Maths 3) 2007]

1. (i) Consider the mapping

$$w = \frac{1}{(z-1)^2}$$

from the z-plane to the w-plane where $z = x + iy$ and $w = u + iv$.

Show that the circle $(x-1)^2 + y^2 = R^2$ in the z-plane maps to the circle $u^2 + v^2 = R^{-4}$ in the w-plane.

(ii) Consider the mapping $w = 1/z$.

For the class of circles

$$(x-1)^2 + (y-1)^2 = r^2 ,$$

show that when $r = \sqrt{2}$ the circle maps to the straight line

$$v = u - 1/2$$

in the w-plane, and that when $r = 1$, the circle maps to the circle

$$(u-1)^2 + (v+1)^2 = 1$$

in the w-plane.

2. Consider the complex function

$$F(z) = \frac{e^{iz}}{z^2+1} .$$

(i) Find the two poles of $F(z)$.

(ii) Find the residues of $F(z)$ at these two poles.

(iii) By considering the semi-circular contour of radius R in the upper half-plane and taking the limit as $R \to \infty$, show that

$$\int_0^\infty \frac{\cos x}{1+x^2}\, dx = \frac{\pi}{2e} .$$

PLEASE TURN OVER

3. Consider a circular contour $\mathcal{C}$ taken as the unit circle $z = e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$.

Convert the integral

$$I = \int_0^{2\pi} \frac{d\theta}{3 + \cos\theta}$$

to the complex integral around $\mathcal{C}$

$$I = \frac{2}{i} \oint_{\mathcal{C}} \frac{dz}{z^2 + 6z + 1},$$

and hence use the Residue Theorem to show that

$$I = \pi / \sqrt{2}.$$

4. If $\overline{f}(\omega)$ is the Fourier transform of $f(t)$, prove Parseval's equality

$$\int_{-\infty}^{\infty} |\, f(t)\, |^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\, \overline{f}(\omega)\, |^2 \, d\omega .$$

The sinc-function sinc (t) and the tent function $\Lambda(t)$ are defined respectively by

$$\text{sinc}(t) = \frac{\sin(\frac{1}{2}t)}{\frac{1}{2}t};$$

$$\Lambda(t) = \begin{cases} 1 + t, & -1 \leq t \leq 0, \\ 1 - t, & 0 \leq t \leq 1. \end{cases}$$

Show that

(i) $$\overline{\Lambda}(\omega) = \text{sinc}^2(\omega);$$

(ii) $$\int_{-\infty}^{\infty} \text{sinc}^4(\omega)\, d\omega = \frac{4\pi}{3}.$$

5. $\overline{f}(s) = \mathcal{L}\{f(t)\}$ and $\overline{g}(s) = \mathcal{L}\{g(t)\}$ are the Laplace transforms of two functions $f(t)$ and $g(t)$ respectively. The convolution of $f(t)$ with $g(t)$ is defined as

$$f * g = \int_0^t f(u)\, g(t-u)\, du \, .$$

(i) Prove the Laplace convolution theorem

$$\mathcal{L}\,(f * g) = \overline{f}(s)\overline{g}(s)$$

by using double integration. Sketch the region of integration.

(ii) Hence, or otherwise, show that

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}\, t \,\sin t \, .$$

You may assume that

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1} \qquad \text{and} \qquad \mathcal{L}\{\sin t\} = \frac{1}{s^2+1} \, .$$

6. A second-order ordinary differential equation, with initial values, takes the form

$$\ddot{x} + 4\dot{x} + 5x = f(t) \, ,$$

where $f(t)$ is a driving force. At $t = 0$, $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$.

(i) When $x_0 = \dot{x}_0 = 0$, use the Laplace convolution theorem to show that the solution of the differential equation can be expressed as

$$x(t) = \int_0^t e^{-2u} \sin u \, f(t-u) du \, . \qquad (*)$$

(ii) When $x_0 = 1$ and $\dot{x}_0 = -2$, show that the solution is comprised of two parts, the first being the integral in (*) and the second being $e^{-2t}\cos t$.

PLEASE TURN OVER

7. Consider a two-dimensional region $\mathcal{R}$ bounded by a closed piecewise-smooth curve C. Using Green's Theorem in a plane, choose the components of a vector field $\boldsymbol{u}(x, y)$ in terms of $P(x, y)$ and $Q(x, y)$ to prove the two-dimensional form of the Divergence Theorem

$$\oint_C \boldsymbol{u} \cdot \boldsymbol{n}\, ds = \iint_{\mathcal{R}} \operatorname{div} \boldsymbol{u}\, dx\, dy \tag{1}$$

where $\boldsymbol{n}$ is a unit vector normal to C.

Suppose

$$\boldsymbol{u}(x, y) \quad \text{is given by}$$

$$\boldsymbol{u}(x, y) = \frac{1}{2} x^2 g'(y) \boldsymbol{i} + x\, g(y) \boldsymbol{j}$$

where $g = g(y)$ is a function with the property $g(0) = 0$.

The region $\mathcal{R}$ lies in the first quadrant and is bounded by the x-axis, the curve $y = x^2$ and the line $x = 1$.

(i) Sketch the region $\mathcal{R}$;

(ii) Evaluate the double integral on the right hand side of (1) to show that

$$\oint_C \boldsymbol{u} \cdot \boldsymbol{n}\, ds = \int_0^1 g(\theta)\, d\theta ,$$

where $\theta = x^2$.

Hint: Green's Theorem in a plane states that for any differentiable functions $P(x, y)$ *and* $Q(x, y)$

$$\oint_C [P(x, y)\, dx + Q(x, y)\, dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

where $\mathcal{R}$ *is a region in the plane and* C *is the counter-clockwise closed curve surrounding* $\mathcal{R}$.

8. (i) The integral

$$\int\int_{\mathcal{R}} \frac{xy}{x^2+y^2}\, dx\, dy$$

is to be evaluated over the region $\mathcal{R}$ bounded by the curves $y = 0,\ y = x$, $x^2 - y^2 = 1$ and $xy = 1$.

Sketch the region $\mathcal{R}$. Find the values of the new variables $u = xy$ and $v = x^2 - y^2$ on each part of the boundary of $\mathcal{R}$ and then evaluate the integral by first changing to the new variables.

(ii) Find the position of the centre of mass of area $\mathcal{S}$ which is the sector of the circle $x^2 + y^2 = 1$ between the lines $x = 0$ and $y = x$ with $x \geq 0$.

Note, the centre of mass is defined by

$$\overline{x} = \frac{1}{A}\int\int_{\mathcal{S}} x\, dx\, dy\ , \quad \overline{y} = \frac{1}{A}\int\int_{\mathcal{S}} y\, dx\, dy$$

where $A = \displaystyle\int\int_{\mathcal{S}} dx\, dy$ is the area of the sector $\mathcal{S}$.

9. A vector field $\boldsymbol{E}(x,\ y,\ z)$ is given by

$$\boldsymbol{E} = e^{ax}\cos by\, \boldsymbol{i} + e^{-x}\sin y\, \boldsymbol{j} + z\, \boldsymbol{k}\ .$$

Find curl $\boldsymbol{E}$ and find the values of a and b for which there exists a scalar field $\phi(x,\ y,\ z)$ such that $\boldsymbol{E} = \operatorname{grad}\phi$. Find $\phi(x,\ y,\ z)$.

Calculate div $\boldsymbol{E}$ and show that for the values of a and b found above

$$\operatorname{div}\boldsymbol{E} = 1\ .$$

Show also that if $\boldsymbol{F} = \operatorname{curl}\boldsymbol{B} + \boldsymbol{E}$, then for these values of a and b and any differentiable vector field $\boldsymbol{B}$ we also have

$$\operatorname{div}\boldsymbol{F} = 1\ .$$

PLEASE TURN OVER

10. Green's theorem in the plane states that

$$\oint_C (f\,dx + g\,dy) = \iint_{\mathcal{R}} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\,dy$$

where C is the counter-clockwise boundary of $\mathcal{R}$.

(i) Use this theorem to write down a condition for the line integral

$$\int_{C_1} (f\,dx + g\,dy)$$

to be independent of the path C_1 joining the two fixed points A and B.

(ii) Hence show that the line integral

$$\int_{C_1} \left[(2e^{2x} \sin y \ \cos y)\,dx + e^{2x}(\cos^2 y - \sin^2 y)\,dy \right]$$

is independent of the path C_1 joining the points $(0,\,0)$ and $(1,\,\pi/4)$ and evaluate it.

(iii) Consider the line integral

$$\oint_C (f\,dx + g\,dy)$$

where

$$f = \frac{-y}{x^2+y^2}\,, \quad g = \frac{x}{x^2+y^2}$$

and C is the counter-clockwise circle centre 0 and radius 1.

Show that

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

everywhere except at the point $(0,\,0)$.

Show that $\oint_C (f\,dx + g\,dy) = 2\pi$.

How do you reconcile this result with Green's theorem in the plane?

Hint: use polar coordinates $x = r\cos\theta,\ y = r\sin\theta$ *to evaluate the line integral.*

11. Consider the discrete random variables X and Y with joint probability mass function $p(x, y) = \mathrm{P}(X = x, Y = y)$ given by the table below.

	$p(x,y)$	y = 0	1	2
x	0	0.42	0.12	0.06
	1	0.30	0.04	0.06

(i) Find the marginal distribution of X and the marginal distribution of Y.

(ii) Find $\mathrm{E}(X)$ and $\mathrm{E}(Y)$.

(iii) Find $\mathrm{P}(X = 1 \mid Y > 0)$.

(iv) Are X and Y correlated? Give your reasoning.

(v) Are X and Y independent? Give your reasoning.

12. Consider a shipment of 8 generators. They produce the following voltages:

$$5.3, \quad 5.6, \quad 4.5, \quad 6.0, \quad 5.0, \quad 5.1, \quad 5.8, \quad 5.9 .$$

Assume that these are independent samples from an $N(\mu, \sigma^2)$ distribution.

(i) Find the sample mean, $\bar{x}$.

(ii) Find the sample variance s^2 and the sample standard deviation s.

(iii) Find a 95% confidence interval for the mean μ.

(iv) Perform a t-test at the 5% level for the hypothesis

$$H_0 : \mu = 5 \quad \text{against} \quad H_1 : \mu \neq 5.$$

END OF PAPER

M A T H E M A T I C S   D E P A R T M E N T

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\,,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots\,,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots\,,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\,D^{n-1}g + \ldots + \tbinom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\;g\,.$$

(b) Taylor's expansion of $f(x)$ about $x=a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a+\theta h)/(n+1)!$, $0<\theta<1$.

(c) Taylor's expansion of $f(x,\,y)$ about $(a,\,b)$:

$$f(a+h,\,b+k) = f(a,\,b) + [hf_x + kf_y]_{a.b} + 1/2!\,\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a.b} + \ldots$$

(d) Partial differentiation of $f(x,\,y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u,\,v)$, $y = y(u,\,v)$, then $f = F(u,\,v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,,\quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\,y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let $(a,\,b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D>0$ and $f_{xx}(a,\,b) < 0$, then $(a,\,b)$ is a maximum;
If $D>0$ and $f_{xx}(a,\,b) > 0$, then $(a,\,b)$ is a minimum;
If $D<0$ then $(a,\,b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\,y)dx + Q(x,\,y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

1. Probabilities for events

For events A, B, and C $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\quad P(A)\,/\,P(\overline{A})$

Conditional probability $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\,P(B \mid A)\,P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) = \dfrac{P(A)\,P(B \mid A)}{P(A)\,P(B \mid A) + P(\overline{A})\,P(B \mid \overline{A})}$

A and B are independent if $\quad P(B \mid A) = P(B)$

A, B, and C are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable X is called the probability mass function (pmf) and is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$, so $E(X^2) = \sum_x x^2 p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \operatorname{var}(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$

Sample variance $\quad s^2 = \dfrac{1}{n-1}\left\{ \sum_k x_k^2 - \dfrac{1}{n}\left(\sum_j x_j\right)^2 \right\}$ estimates σ^2

Standard deviation $\quad \operatorname{sd}(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

Skewness $\quad \beta_1 = E\left(\dfrac{X - \mu}{\sigma}\right)^3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X - \mu}{\sigma}\right)^4 - 3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\quad \widetilde{x}$ or x_{med}. Half the sample values are smaller and half larger

If the sample values $x_1, \ldots, x_n$ are ordered as $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$,
then $\widetilde{x} = x_{(\frac{n+1}{2})}$ if n is odd, and $\widetilde{x} = \frac{1}{2}\left(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})}\right)$ if n is even

α-quantile $Q(\alpha)$ is such that $P(X \leq Q(\alpha)) = \alpha$

Sample α-quantile $\widehat{Q}(\alpha)$ Proportion α of the data values are smaller

Lower quartile Q1 $= \widehat{Q}(0.25)$ one quarter are smaller

Upper quartile Q3 $= \widehat{Q}(0.75)$ three quarters are smaller

Sample median $\widetilde{x} = \widehat{Q}(0.5)$ estimates the population median $Q(0.5)$

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $\quad F(x) = P(X \leq x) = \displaystyle\int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The probability density function (pdf) $\quad f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$E(X) = \mu = \displaystyle\int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x\,,\quad \mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2,$ where $E(X^2) = \displaystyle\int_{-\infty}^{\infty} x^2\, f(x)\mathrm{d}x$

4. Discrete probability distributions

Discrete Uniform *Uniform* (n)

$$p_x = \frac{1}{n} \quad (x = 1, 2, \ldots, n) \qquad \mu = (n+1)/2,\ \sigma^2 = (n^2-1)/12$$

Binomial distribution *Binomial* (n, θ)

$$p_x = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n) \qquad \mu = n\theta\,,\ \sigma^2 = n\theta(1-\theta)$$

Poisson distribution *Poisson* (λ)

$$p_x = \frac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots) \ \text{(with } \lambda > 0) \qquad \mu = \lambda\,,\ \sigma^2 = \lambda$$

Geometric distribution *Geometric* (θ)

$$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots) \qquad \mu = \frac{1}{\theta}\,,\ \sigma^2 = \frac{1-\theta}{\theta^2}$$

5. Continuous probability distributions

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & \text{(otherwise)}. \end{cases} \qquad \mu = (\alpha+\beta)/2\,,\ \sigma^2 = (\beta-\alpha)^2/12$$

Exponential distribution *Exponential* (λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \leq 0). \end{cases} \qquad \mu = 1/\lambda\,,\ \sigma^2 = 1/\lambda^2$$

Normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} \quad (-\infty < x < \infty), \quad E(X) = \mu, \quad \mathrm{var}(X) = \sigma^2$$

Standard normal distribution $N(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t	$R(t) = P(T > t)$
The failure rate or hazard function	$h(t) = f(t)/R(t)$
The cumulative hazard function	$H(t) = \int_0^t h(t_0)\,\mathrm{d}t_0 = -\ln\{R(t)\}$
The Weibull(α, β) distribution has	$H(t) = \beta t^\alpha$

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P($"device i operates"$)$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$

A system of devices in parallel operates if any device operates

$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1 - R_1)(1 - R_2) \cdots (1 - R_k)$

8. Covariance and correlation

The covariance of X and Y $\quad \mathrm{cov}(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance	$s_{xy} = \dfrac{1}{n-1} S_{xy}$	estimates $\mathrm{cov}(X,Y)$
Correlation coefficient	$\rho = \mathrm{corr}(X,Y) = \dfrac{\mathrm{cov}(X,Y)}{\mathrm{sd}(X) \cdot \mathrm{sd}(Y)}$	
Sample correlation coefficient	$r = \dfrac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$	estimates ρ

9. Sums of random variables

$$E(X+Y) = E(X) + E(Y)$$

$$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2\,\text{cov}(X,Y)$$

$$\text{cov}(aX+bY,\ cX+dY) = (ac)\,\text{var}(X) + (bd)\,\text{var}(Y) + (ad+bc)\,\text{cov}(X,Y)$$

If X is $N(\mu_1, \sigma_1^2)$, Y is $N(\mu_2, \sigma_2^2)$, and $\text{cov}(X,Y) = c$, then $X+Y$ is $N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t $\quad \text{bias}(t) = E(T) - \theta$

Standard error of t $\quad \text{se}(t) = \text{sd}(T)$

Mean square error of t $\quad \text{MSE}(t) = E\{(T-\theta)^2\} = \{\text{se}(t)\}^2 + \{\text{bias}(t)\}^2$

If $\overline{x}$ estimates μ, then $\text{bias}(\overline{x}) = 0$, $\text{se}(\overline{x}) = \sigma/\sqrt{n}$, $\text{MSE}(\overline{x}) = \sigma^2/n$, $\widehat{\text{se}}(\overline{x}) = s/\sqrt{n}$

Central limit property $\quad$ If n is fairly large, $\overline{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.

For a random sample $x_1, x_2, \ldots, x_n$

$$\ell(\theta;\, x_1, x_2, \ldots, x_n) = P(X_1 = x_1 \,|\, \theta) \cdots P(X_n = x_n \,|\, \theta) \qquad \text{(discrete distribution)}$$

$$\ell(\theta;\, x_1, x_2, \ldots, x_n) = f(x_1 \,|\, \theta)\, f(x_2 \,|\, \theta) \cdots f(x_n \,|\, \theta) \qquad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\widehat{\theta}$ for which the likelihood is a maximum

12. Confidence intervals

If $x_1, x_2, \ldots, x_n$ are a random sample from $N(\mu, \sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\overline{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \overline{x} + 1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0 = t_{n-1,0.05}$

The 95% confidence interval for μ is $(\overline{x} - t_0\frac{s}{\sqrt{n}},\ \overline{x} + t_0\frac{s}{\sqrt{n}})$

13. Standard normal table — Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.999
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table — Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

m	p= 0.10	0.05	0.02	0.01	m	p= 0.10	0.05	0.02	0.01
1	6.31	12.71	31.82	63.66	9	1.83	2.26	2.82	3.25
2	2.92	4.30	6.96	9.92	10	1.81	2.23	2.76	3.17
3	2.35	3.18	4.54	5.84	12	1.78	2.18	2.68	3.05
4	2.13	2.78	3.75	4.60	15	1.75	2.13	2.60	2.95
5	2.02	2.57	3.36	4.03	20	1.72	2.09	2.53	2.85
6	1.94	2.45	3.14	3.71	25	1.71	2.06	2.48	2.78
7	1.89	2.36	3.00	3.50	40	1.68	2.02	2.42	2.70
8	1.86	2.31	2.90	3.36	∞	1.645	1.96	2.326	2.576

15. Chi-squared table — Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and p = .995, .975, *etc*

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 \;=\; \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$$p_{x\bullet} \;=\; \sum_y p_{xy} \quad \text{and} \quad P(X = x \,|\, Y = y) \;=\; \frac{p_{xy}}{p_{\bullet y}}$$

Continuous distribution

Joint cdf $\quad F(x,y) \;=\; P(\{X \le x\} \cap \{Y \le y\}) \;=\; \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0\,, y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0$

Joint pdf $\quad f(x,y) \;=\; \frac{\mathrm{d}^2 F(x,y)}{\mathrm{d}x\,\mathrm{d}y}$

Marginal pdf of X $\quad f_X(x) \;=\; \int_{-\infty}^{\infty} f(x, y_0)\,\mathrm{d}y_0$

Conditional pdf of X given $Y = y$ $\quad f_{X|Y}(x|y) \;=\; \frac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. Linear regression

To fit the linear regression model $y = \alpha + \beta x$ by $\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is $\quad \widehat{\alpha} \;=\; \overline{y} - \overline{x}\widehat{\beta}\,, \quad \widehat{\beta} \;=\; \frac{S_{xy}}{S_{xx}}$

The residual sum of squares $\quad \text{RSS} \;=\; S_{yy} - \frac{S_{xy}^2}{S_{xx}}$

$\widehat{\sigma^2} = \frac{\text{RSS}}{n-2} \qquad \frac{n-2}{\sigma^2}\,\widehat{\sigma^2}$ is from χ^2_{n-2}

$$E(\widehat{\alpha}) \;=\; \alpha\,, \quad E(\widehat{\beta}) \;=\; \beta\,,$$

$$\operatorname{var}(\widehat{\alpha}) \;=\; \frac{\sum x_i^2}{n\,S_{xx}}\sigma^2\,, \quad \operatorname{var}(\widehat{\beta}) \;=\; \frac{\sigma^2}{S_{xx}}\,, \quad \operatorname{cov}(\widehat{\alpha}, \widehat{\beta}) \;=\; -\frac{\overline{x}}{S_{xx}}\,\sigma^2$$

$$\widehat{y}_x = \widehat{\alpha} + \widehat{\beta} x\,, \qquad E(\widehat{y}_x) = \alpha + \beta x\,, \qquad \operatorname{var}(\widehat{y}_x) \;=\; \left\{\frac{1}{n} + \frac{(x - \overline{x})^2}{S_{xx}}\right\}\sigma^2$$

$\frac{\widehat{\alpha} - \alpha}{\widehat{\text{se}}\,(\widehat{\alpha})}\,, \qquad \frac{\widehat{\beta} - \beta}{\widehat{\text{se}}\,(\widehat{\beta})}\,, \qquad \frac{\widehat{y}_x - \alpha - \beta\,x}{\widehat{\text{se}}\,(\widehat{y}_x)}$ are each from t_{n-2}

EXAMINATION QUESTIONS/SOLUTIONS 2006-07

Course: Core

Question: 8 SOL

Marks & seen/unseen

Parts

a)

$$w = \frac{1}{(z-1)^2} = \frac{[(x-1)-iy]^2}{[(x-1)^2+y^2]^2} = u + iv$$

$$u = \frac{(x-1)^2-y^2}{[(x-1)^2+y^2]^2} \quad \& \quad v = \frac{-2y(x-1)}{[(x-1)^2+y^2]^2}$$

$$\Rightarrow u^2+v^2 = \frac{1}{[(x-1)^2+y^2]^2} = \frac{1}{(R^2)^2} \quad ✓$$

6

circle in the w-plane with radius R^{-2}

(Alternatively, note the student could also correctly answer:

if $(x-1)^2+y^2 = R^2$

$\Rightarrow |z-1| = R$

& hence $w = \frac{1}{(z-1)^2} \Rightarrow |w|^2 = \frac{1}{|z-1|^2} = 1/R^4$.)

Setter's initials: DB

Checker's initials: MH

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course CORE
Question 9 b Soln		Marks & seen/unseen

Parts

$$w = 1/z = \frac{x - iy}{x^2 + y^2}$$

$$\Rightarrow u = \frac{x}{x^2+y^2} \quad \& \quad v = \frac{-y}{x^2+y^2}$$

Note also $z = 1/w$

$$\Rightarrow x = \frac{u}{u^2+v^2} \quad \& \quad y = \frac{-v}{u^2+v^2}$$

So $r^2 = (x-1)^2 + (y-1)^2$

$$= \frac{(u - u^2 - v^2)^2 + (v + u^2 + v^2)^2}{(u^2+v^2)^2}$$ 4

$$\Rightarrow (u^2+v^2)^2 r^2 = (u^2+v^2)\left[1 - 2u + 2v\right] + 2(u^2+v^2)^2$$ ✓

So if $r^2 = 2 \Rightarrow 1 = 2u - 2v$

$\Rightarrow v = u - 1/2$, a straight line. 5

But if $r^2 = 1 \Rightarrow (u^2+v^2)(1-2u+2v) + (u^2+v^2)^2 = 0$

$\Rightarrow 1 - 2u + 2v + u^2 + v^2 = 0$ 5

$\Rightarrow (u-1)^2 + (v+1)^2 = 1$ circle of rad $=1$ centered @ $(1,-1)$

Setter's initials: DB | Checker's initials: MH | Page number: 4

AOG

EE I(3) 1

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course CORE
Question 8b SOL^n		Marks & seen/unseen
Parts	N.B. The student may go the opposite direction: $$u - 1/2 = v \Rightarrow \frac{x}{x^2+y^2} - \frac{1}{2} = \frac{-y}{x^2+y^2}$$ $$\Rightarrow (x-1)^2 + (y-1)^2 = 2 \ldots$$ This should also be given full marks.	Total $\boxed{20}$
	Setter's initials	Checker's initials MH.	Page number 5

2/1

SOLUTION

(i) Poles at $z^2+1=0$

ie $z=\pm i$ (2)

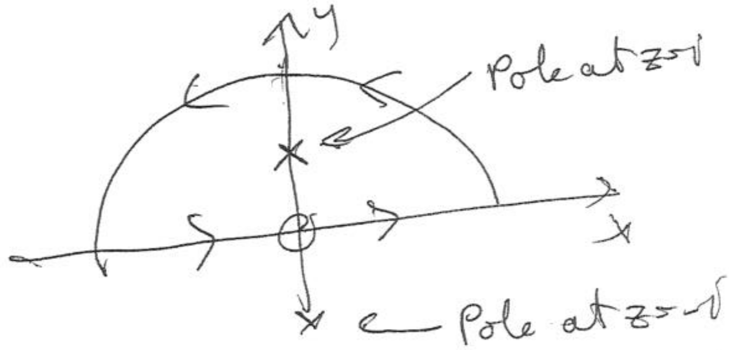
(ii) Residue at $z=i$ is

$$R_1 = \lim_{z\to i} \frac{(z-i)\,e^{iz}}{z^2+1} = \frac{e^{-1}}{2i}$$ (2)

Residue at $z=-i$ is

$$R_2 = \lim_{z\to -i} \frac{(z+i)\,e^{iz}}{z^2+1} = -\frac{e^{1}}{2i}$$ (2)

(iii) Contour C



$$\therefore \oint_C \frac{e^{iz}}{z^2+1}\,dz = 2\pi i \sum(\text{residues within } C)$$

Cauchy's theorem

$$= 2\pi i \frac{e^{-1}}{2i} = \frac{\pi}{e}$$ (6)

As $R\to\infty$ $e^{iz}\to 0$ provided z is in upper half-plane.

Similarly $\frac{1}{z^2+1} \to 0$

RLJ

2/2
SOLUTION

EE II (3) 2

Hence by Jordan's Lemma

integral over semicircle $\to 0$ as $R \to \infty$ (3)

$$\therefore \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx = \frac{\pi}{e}$$

Hence $\int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{x^2+1}\,dx = \frac{\pi}{e}$ (2)

$\frac{\sin x}{x^2+1}$ is odd and integral is over symmetric range hence integral is zero

$\frac{\cos x}{x^2+1}$ is even and integral is over symmetric range hence we can

write $2\int_0^{\infty} \frac{\cos x}{x^2+1}\,dx = \frac{\pi}{e}$ (3)

Result follows

20

RLJ, J.D. Gibbon.

MARKS

$$I = \int_0^{2\pi} \frac{d\theta}{3+\cos\theta} \qquad \cos\theta = \tfrac{1}{2}(e^{i\theta}+e^{-i\theta}) = \tfrac{1}{2}\frac{(z^2+1)}{z}$$ 2

C: $z = e^{i\theta}$ $\qquad dz = i e^{i\theta} d\theta = i z\, d\theta$ 2

$$\therefore \quad I = \frac{1}{i} \oint_C \frac{1}{z} \cdot \frac{dz}{3 + \frac{z^2+1}{2z}}$$ 3

$$= \frac{2}{i} \oint_C \frac{dz}{z^2+6z+1}$$ 3

$z^2+6z+1=0$ has roots at $z_{1,2} = -3 \pm 2\sqrt{2}$ 2

Both are -ve: one lies in C & one outside C.

$z_1 = -3+2\sqrt{2}$ $\qquad z_2 = -3-2\sqrt{2}$ 2

Thus only z_1 counts.

The residue of $\frac{1}{z^2+6z+1}$ at z_1 is $\frac{1}{z_1-z_2}$

$$= \frac{1}{4\sqrt{2}}$$ 3

$$\therefore \quad I = \frac{2}{i} \cdot 2\pi i \times \frac{1}{4\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$ 3

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\bar{f}(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt\right)\left(\int_{-\infty}^{\infty} e^{+i\omega\tau} f^*(\tau)\,d\tau\right)d\omega$$

$$= \int_{-\infty}^{\infty} f(t)\left\{\int_{-\infty}^{\infty}\underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(\tau-t)}d\omega\right)}_{\delta(\tau-t)} f^*(\tau)\,d\tau\right\}dt$$

$$= \int_{-\infty}^{\infty} f(t)\left(\int_{-\infty}^{\infty}\delta(\tau-t)f^*(\tau)\,d\tau\right)dt = \int_{-\infty}^{\infty}|f(t)|^2 dt \qquad 8$$

i) $$\bar{\Lambda}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\Lambda(t)\,dt = \int_{-1}^{0}(1+t)e^{-i\omega t}dt + \int_0^1(1-t)e^{-i\omega t}dt$$

$$= -\frac{1}{i\omega}\left\{\int_{-1}^{0}(1+t)\,d(e^{-i\omega t}) + \int_0^1(1-t)\,d(e^{-i\omega t})\right\}$$

$$= \frac{i}{\omega}\left\{\left[e^{-i\omega t}(1+t)\right]_{-1}^{0} - \int_{-1}^{0}e^{-i\omega t}dt + \left[e^{-i\omega t}(1-t)\right]_0^1 + \int_0^1 e^{-i\omega t}dt\right\}$$

$$= \frac{1}{\omega^2}\left(2 - e^{i\omega} - e^{-i\omega}\right) = \frac{2-2\cos\omega}{\omega^2} = \frac{4\sin^2\frac{1}{2}\omega}{\omega^2} \qquad 6$$

$$= \operatorname{sinc}^2\omega$$

ii) $$\int_{-\infty}^{\infty}\operatorname{sinc}^4\omega\,d\omega = \int_{-\infty}^{\infty}|\bar{\Lambda}(\omega)|^2 d\omega \quad \text{by (i)}$$

$$= 2\pi\int_{-\infty}^{\infty}|\Lambda(t)|^2 dt \quad \text{Parseval}$$

$$= 2\pi\left\{\int_{-1}^{0}(1+t)^2 dt + \int_0^1(1-t)^2 dt\right.$$

$$= 2\pi\left\{\left[t+t^2+\tfrac{1}{3}t^3\right]_{-1}^{0} + \left[t-t^2+\tfrac{1}{3}t^3\right]_0^1\right\} \qquad 6$$

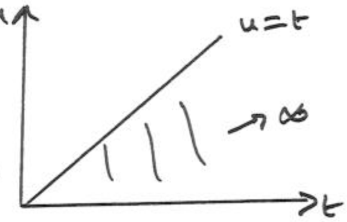
$$= 4\pi/3.$$

Setter: JDG. Checker: ADG 17/11/06

C8-5 SOLUTION EEII(3) (5) MARKS

i) $\mathcal{L}(f*g) = \int_0^\infty e^{-st}\left(\int_0^t f(u)g(t-u)\,du\right)dt$ 4

$= \int_0^\infty f(u)\left(\int_{t=u}^{t=\infty} e^{-st} g(t-u)\,dt\right)du$ 2 (pic)



For the inner integral define $\theta = t-u$

$\mathcal{L}(f*g) = \int_0^\infty f(u)\left(\int_{\theta=0}^{\theta=\infty} e^{-s(\theta+u)} g(\theta)\,d\theta\right)du$ 2

$= \int_0^\infty f(u)e^{-su}\,du \int_0^\infty g(\theta)e^{-s\theta}\,d\theta = \bar{f}(s)\bar{g}(s)$ 2

ii) Write $\frac{s}{(s^2+1)^2} = \frac{s}{s^2+1}\cdot\frac{1}{s^2+1} \equiv \bar{f}(s)\,\bar{g}(s)$ 2

$\bar{f}(s) = \frac{s}{s^2+1} \Rightarrow f(t) = \cos t$ 2

$\bar{g}(s) = \frac{1}{s^2+1} \Rightarrow g(t) = \sin t$ 2

By the convolution theorem

$\mathcal{L}^{-1}\left(\bar{f}(s)\bar{g}(s)\right) = f*g = \int_0^t \cos u \sin(t-u)\,du$

$= \sin t\int_0^t \cos^2 u\,du - \cos t\int_0^t \cos u \sin u\,du$

$= \frac{1}{2}\sin t\int_0^t (1+\cos 2u)\,du - \cos t\int \sin u\, d(\sin u)$

$= \frac{1}{2}\sin t\left[u + \frac{1}{2}\sin 2u\right]_0^t - \frac{1}{2}\cos t\left[\sin^2 u\right]_0^t$

$= \frac{1}{2}\sin t\left[t + \sin t\cos t\right] - \frac{1}{2}\cos t\sin^2 t$ 4

$= \frac{1}{2}t\sin t$

SETTER: JDG CHECKER: AOG 17/11/06

$$\ddot{x} + 4\dot{x} + 5x = f(t)\,; \quad x_0 = 0 \quad \dot{x}_0 = 0$$

$$\mathcal{L}(\dot{x}) = s\bar{x}(s) - x_0$$

Tables.

$$\mathcal{L}(\ddot{x}) = s^2\bar{x}(s) - s x_0 - \dot{x}_0$$

(i) $\therefore \left(s^2\bar{x}(s) - s x_0 - \dot{x}_0\right) + 4\left(s\bar{x}(s) - x_0\right) + 5\bar{x}(s) = \bar{f}(s)$ 2

$$\therefore \quad \bar{x}(s) = \frac{\bar{f}(s)}{s^2 + 4s + 5}$$

$$= \frac{\bar{f}(s)}{(s+2)^2 + 1}$$ completing the square 4

$$\bar{x}(s) = \bar{f}(s)\,\bar{g}(s)$$

where $\bar{g}(s) = \dfrac{1}{(s+2)^2 + 1}$ 4

Firstly $\bar{f}(s) \to f = f(t)$

$$\bar{g}(s) = \frac{1}{(s+2)^2 + 1} \to g(t) = e^{-2t}\sin t$$ 3

Shift Thm

Convolution $\to$ $x(t) = \int_0^t e^{-2u}\sin u\, f(t-u)\,du$. 2

(ii) When the ICs are changed to $x_0 = 1$, $\dot{x}_0 = -2$

we have

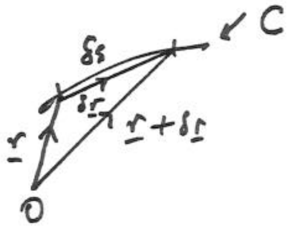
$$\bar{x}(s) = \frac{\bar{f}(s)}{(s+2)^2+1} + \frac{s+2}{(s+2)^2+1}$$ 5

By shift rule $\mathcal{L}^{-1}\left\{\dfrac{s+2}{(s+2)^2+1}\right\} = e^{-2t}\cos t$

which is the additional term

Setter: JDG Checker: AOG 17/11/06

Unit tangent vector $\hat{\underline{t}} = \frac{d\underline{r}}{ds} = \underline{\hat{i}}\frac{dx}{ds} + \underline{\hat{j}}\frac{dy}{ds}$



Unit normal vector $\underline{n}\cdot\hat{\underline{t}} = 0$

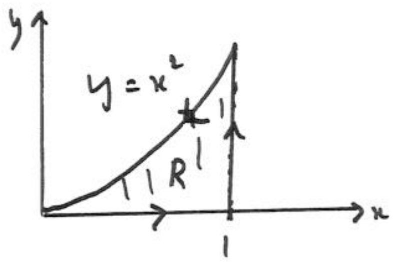
Thus write $\underline{n} = \pm\left(\underline{\hat{i}}\frac{dy}{ds} - \underline{\hat{j}}\frac{dx}{ds}\right)$ 2

Choose $\underline{u} = \underline{\hat{i}}\,Q - \underline{\hat{j}}\,P$ 2

$\operatorname{div}\underline{u} = Q_x - P_y$; $\underline{u}\cdot\underline{n} = Q\frac{dy}{ds} + P\frac{dx}{ds}$ 2+2

$\therefore$ G.T. $\rightarrow$ $\oint_C (\underline{u}\cdot\underline{n})\,ds = \iint_R \operatorname{div}\underline{u}\,dx\,dy$ 2

i)



2

ii) $\oint \underline{u}\cdot\underline{n}\,ds = \iint_R \operatorname{div}\underline{u}\,dx\,dy$

$\operatorname{div}\underline{u} = 2x\,g'(y)$

$$\therefore \oint \underline{u}\cdot\underline{n}\,ds = 2\iint_R x\,g'(y)\,dx\,dy$$

$$= 2\int_0^1 x\left(\int_0^{x^2} g'(y)\,dy\right)dx$$

$$= 2\int_0^1 x\,g(x^2)\,dx \qquad g(0) = 0$$

$$= \int_0^1 g(\theta)\,d\theta \qquad \theta = x^2$$ 8

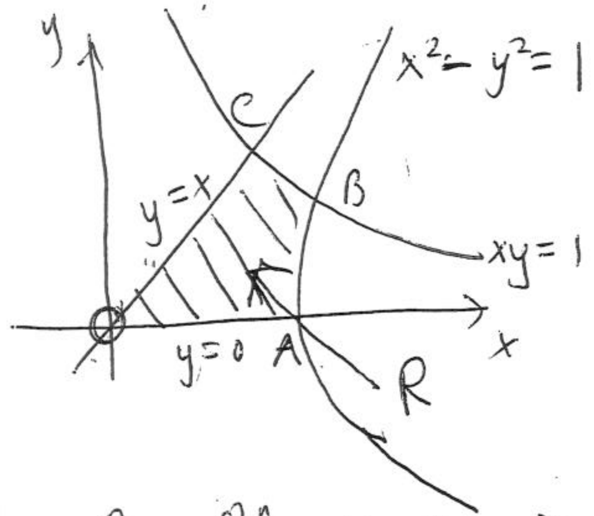
EE II (3) (8)

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course
Question C3		Marks & seen/unseen

Parts

(a)

All unseen



2

On OA $y=0 \Rightarrow u=0$

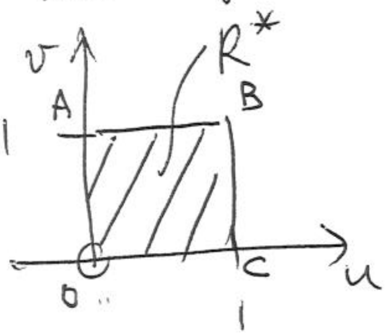
On AB $x^2-y^2=1 \Rightarrow v=1$

On BC $xy=1 \Rightarrow u=1$

On CO $y=x \Rightarrow v=0$

2

New region of integration



2

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix} = -2(y^2+x^2)$$

2

$$\therefore \iint_R \frac{xy}{x^2+y^2}\,dx\,dy = \int_{v=0}^{1}\int_{u=0}^{1} \frac{u}{\cancel{x^2+y^2}} \, \frac{1}{2\cancel{(x^2+y^2)}}\,du\,dv$$

$$= \tfrac{1}{2}$$

2

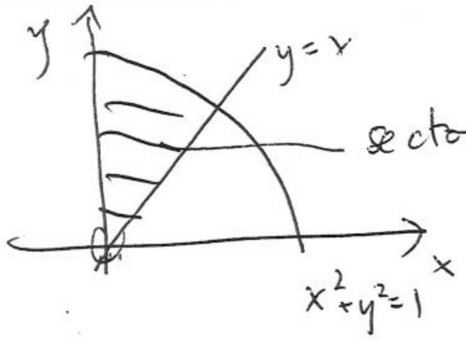
p1/2 Setter's initials RLJ Checker's initials DB Page number

EE II(3) (8)

	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course
Question 3	Sol^n	Marks & seen/unseen

Parts

(b)



$A =$ Area of sector $= \frac{1}{8}$ Area of circle $\quad$ 2

$= \frac{1}{8}\pi.$

$\bar{x} = \frac{1}{A} \iint_S x \, dx\, dy$ $\quad$ 2

Change to polars. Remember $J = r$

$\therefore \bar{x} = \left(\frac{1}{8}\pi\right)^{-1} \int_{r=0}^{1} \int_{\theta=\pi/4}^{\pi/2} r \cos\theta \, r \, dr \, d\theta$

$= \left(\frac{1}{8}\pi\right)^{-1} \left[\frac{r^3}{3}\right]_0^1 \left[\sin\theta\right]_{\pi/4}^{\pi/2}$ $\quad$ 3

$= \frac{8}{3\pi}\left(1 - \frac{1}{\sqrt{2}}\right)$

$\bar{y} = \left(\frac{1}{8}\pi\right)^{-1} \int_{r=0}^{1} \int_{\theta=\pi/4}^{\pi/2} r \sin\theta \, r \, dr \, d\theta$ $\quad$ 3

$= \left(\frac{1}{8}\pi\right)^{-1} \left[\frac{r^3}{3}\right]_0^1 \left[-\cos\theta\right]_{\frac{\pi}{4}}^{\pi/2}$

$= \frac{8}{3\sqrt{2}\pi}$

Total 20

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P2/2

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	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course
Question C4	sol^n	Marks & seen/unseen

Parts

$$\text{curl } \underline{E} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ e^{ax}\cos by & e^{-x}\sin y & z \end{vmatrix}$$

all unseen

$$= \underline{i}\,0 + \underline{j}\cdot 0 + \underline{k}(-e^{-x}\sin y + be^{ax}\sin by)$$ 4

~~⟹ a = −1, b = 1 for~~ A scalar field φ exists if curl $\underset{\sim}{E} = 0$ 2

$\Rightarrow a = -1,\ b = 1.$ 2

Then $\frac{\partial \varphi}{\partial x} = e^{-x}\cos y$, $\frac{\partial \varphi}{\partial y} = e^{-x}\sin y$, $\frac{\partial \varphi}{\partial z} = z$

$$\left.\begin{aligned} \Rightarrow \varphi &= -e^{-x}\cos y + f(y,z) \\ \varphi &= -e^{-x}\cos y + g(x,z) \\ \varphi &= \frac{z^2}{2} + h(x,y) \end{aligned}\right\} \Rightarrow \varphi(x,y,z) = -e^{-x}\cos y + \frac{z^2}{2} + C$$ 6

3

$$\text{div } \underline{E} = a e^{ax}\cos by + e^{-x}\cos y + 1 = 1$$

if $a = -1$ and $b = 1$

Finally $\text{div } \underline{F} = \text{div curl } \underset{\sim}{B} + \text{div } \underline{E}$

$$= 0 + 1$$ 3

because div curl $\underline{B} = 0$ for any suitably differentiable $\underline{B}$.

Total 20

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Question C5 SOLUTION | Marks & seen/unseen

Parts

(a) If $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$ at every point in R

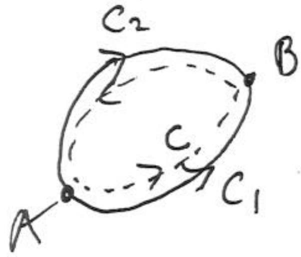
then from G T in Plane we have

$$\oint_C (f\,dx + g\,dy) = 0$$

$$\therefore \int_{C_1} (f\,dx + g\,dy) = \int_{C_2} (f\,dx + g\,dy)$$

Bookwork

3



Rest unseen

(b) $f = 2e^{2x} \sin y \cos y$

$g = e^{2x}(\cos^2 y - \sin^2 y)$

$\frac{\partial f}{\partial y} = 2e^{2x}(\cos^2 y - \sin^2 y)$

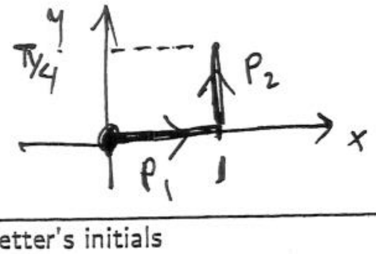
$\frac{\partial g}{\partial x} = 2e^{2x}(\cos^2 y - \sin^2 y)$

4

Thus $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ everywhere

Hence integral is independent of path.

Choose a very simple path.



C_1 is made up of 2 bits P_1 and P_2

(Any other path used correctly should get full credit)

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PT/2

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	EXAMINATION QUESTIONS/SOLUTIONS 2006-07	Course
Question C5	Solution	Marks & seen/unseen
Parts	On P_1 $y=0$, $\therefore dy=0$ and $\sin y=0$	

$$\int_{P_1} = 0$$

On P_2 $x=1$ $\therefore dx=0$ and

$$\int_{P_2} = \int_0^{\pi/4} e^2(\cos^2 y-\sin^2 y)\,dy$$

$$= e^2\int_0^{\pi/4}\cos 2y\,dy$$

$$= e^2\left[\frac{\sin 2y}{2}\right]_0^{\pi/4} = \frac{1}{2}e^2.$$ 4

(c) $f = \dfrac{-y}{x^2+y^2}$, $g = \dfrac{x}{x^2+y^2}$

$$\frac{\partial f}{\partial y} = -\frac{1}{x^2+y^2}+\frac{y.2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$ 4

$$\frac{\partial g}{\partial x} = \frac{1}{x^2+y^2}-\frac{x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Thus $\frac{\partial f}{\partial y}=\frac{\partial g}{\partial x}$ everywhere except at $(0,0)$ where functions and derivatives are undefined.

$$\oint_C (f\,dx+g\,dy) = \int_0^{2\pi}\left[\frac{-\sin\theta}{1}(-\sin\theta)\,d\theta + \frac{\cos\theta}{1}\cos\theta\,d\theta\right] \quad \begin{array}{l} x=\cos\theta \\ y=\sin\theta \end{array}$$ 4

$$= \int_0^{2\pi}(\sin^2\theta+\cos^2\theta)\,d\theta = 2\pi.$$

For G.T. in P to give ~~correct~~ condition which must be satisfied at every point in R. 1

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Total (20)

11

	EXAMINATION SOLUTIONS 2006-07	Course EE2(3)
Question 11		Marks & seen/unseen

Parts

(i)
$P(X=0) = 0.42 + 0.12 + 0.06 = 0.60$ — 1
$P(X=1) = 0.30 + 0.04 + 0.06 = 0.40$ — 1

$P(Y=0) = 0.42 + 0.30 = 0.72$ — 1
$P(Y=1) = 0.12 + 0.04 = 0.16$ — 1
$P(Y=2) = 0.06 + 0.06 = 0.12$ — 1

(ii)
$E(X) = 0 \cdot 0.60 + 1 \cdot 0.40 = 0.40;$ — 2
$E(Y) = 0 \cdot 0.72 + 1 \cdot 0.16 + 2 \cdot 0.12 = 0.40;$ — 3

(iii)

Seen Similarly

$$P(X=1|Y>0) = \frac{P(X=1, Y>0)}{P(Y>0)} = \frac{P(X=1, Y=1) + P(X=1, Y=2)}{P(Y=1) + P(Y=2)} = \frac{0.04 + 0.06}{0.16 + 0.12} = \frac{0.10}{0.28} = \frac{5}{14} (\approx 0.36)$$

3

(iv)
$E(XY) = 1 \cdot 0.04 + 2 \cdot 0.06 = 0.16$
$\text{cov}(X,Y) = E(XY) - E(X)\,E(Y) = 0.16 - 0.16 = 0.$
Hence,

$$\rho = \text{corr}(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{\text{Var}(X) \cdot \text{Var}(Y)}} = 0.$$

By definition this implies that X and Y are uncorrelated. — 4

(v)
X and Y are not independent (i.e. they are dependent) because

$$P(X=1)\,P(Y=2) = 0.40 \cdot 0.12 = 0.048 \neq 0.06 = P(X=1, Y=2).$$

3

(Of course, any other choice besides $X=1$, $Y=2$ can be used to show that X and Y are dependent.)

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(12)

	EXAMINATION SOLUTIONS 2006-07	Course EE2(3)
Question 12		Marks & seen/unseen

Parts

(i) $\bar{x} = \frac{1}{8}(5.3 + 5.6 + 4.5 + 6.0 + 5.0 + 5.1 + 5.8 + 5.9) = 5.4$ — 2

(ii) Let $x_1, \ldots, x_8$ denote the observed values. — Seen Similarly

$$s^2 = \frac{1}{8-1}\left(\sum_{k=1}^{8} x_k^2 - \frac{1}{8}\left(\sum_{j=1}^{8} x_j\right)^2\right) = \frac{1}{7}\left(235.16 - \frac{1}{8}(43.2)^2\right)$$
$$= \frac{1}{7}(235.16 - 233.28) = \frac{1.88}{7} \approx 0.269$$

3

so $s = \sqrt{\frac{1.88}{7}} \approx 0.518.$ — 1

(iii) From the Student t table we find $t_0 = t_{8-1,0.05} = 2.36.$ — 2

The 95% confidence interval for μ is thus

$$(\bar{x} - t_0\frac{s}{\sqrt{8}}, \bar{x} + t_0\frac{s}{\sqrt{8}}) = (5.4 - 2.36\sqrt{\frac{1.88}{56}}, 5.4 + 2.36\sqrt{\frac{1.88}{56}})$$
$$\approx (4.97, 5.83)$$

4

(iv) The test statistic is

$$t = \frac{\bar{x} - 5}{s/\sqrt{8}} = 0.4\sqrt{\frac{56}{1.88}} \approx 2.18.$$

The corresponding quantile t_0 from the Student t distribution is

$$t_{7,0.05} = 2.36.$$

Since $|t| < 2.36$ the hypothesis H_0 is not rejected. — 8

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