E2.8 - MATHS I
(EE - 2nd yr)

UNIVERSITY OF LONDON

[II(3)E 2005]

B.ENG. AND M.ENG. EXAMINATIONS 2005

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART II : MATHEMATICS 3 (ELECTRICAL ENGINEERING)

Wednesday 1st June 2005 2.00 - 5.00 pm

Answer EIGHT questions.

Answers to Section A *questions must be written in a different answer book from answers to* Section B *questions.*

A statistics data sheet is provided.

[*Before starting, please make sure that the paper is complete; there should be 8 pages, with a total of 12 questions. Ask the invigilator for a replacement if your copy is faulty.*]

SECTION A

1. Consider the mapping

$$w = \frac{1}{z-4}$$

from the z-plane ($z = x + iy$) to the w-plane ($w = u + iv$).

(i) Find u and v in terms of x and y.

(ii) Show that the circle in the z-plane

$$(x-4)^2 + y^2 = 5^2$$

maps to a circle centred at (0, 0) and of radius 1/5 in the w-plane.

(iii) Show that the straight line $y = x - 4$ maps to the straight line $v = -u$ in the w-plane.

(iv) To what does the straight line $x = 0$ map in the w-plane?

(v) To what does the line straight line $x = 4$ map in the w-plane?

(vi) Where are the fixed points of this mapping?

2. Consider the contour integral

$$\oint_C \frac{e^{iz}}{(z^2+4)^2}\, dz \,,$$

where the closed contour C consists of a semi-circle in the upper half of the complex plane.

Use the Residue Theorem to show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+4)^2}\, dx = \frac{3\pi}{16e^2} \,.$$

The residue of a complex function $f(z)$ at a pole $z = a$ of multiplicity n is given by

$$\lim_{z\to a} \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} \{(z-a)^n f(z)\} \right].$$

PLEASE TURN OVER

3. Consider the real integral

$$I = \int_0^{2\pi} \frac{d\theta}{3 + \cos\theta}.$$

Taking the contour C as the unit circle $z = e^{i\theta}$, show that

$$I = \frac{2}{i} \oint_C \frac{dz}{z^2 + 6z + 1}.$$

Hence show that

$$I = \frac{2\pi}{\sqrt{8}}.$$

The residue of a complex function $f(z)$ at a pole $z = a$ of multiplicity n is given by

$$\lim_{z \to a} \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} \{(z-a)^n f(z)\} \right].$$

4. The sawtooth function $\Pi(t)$, the tent function $\Lambda(t)$, and the sinc-function $\mathrm{sinc}\,(t)$ are defined respectively by

$$\Pi(t) = \begin{cases} 1, & -1/2 \le t \le 1/2, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Lambda(t) = \begin{cases} 1+t, & -1 \le t \le 0, \\ 1-t, & 0 \le t \le 1, \end{cases}$$

and

$$\mathrm{sinc}\,(t) = \frac{\sin(t/2)}{(t/2)}.$$

Show that the Fourier transforms $\overline{\Pi}(\omega)$ and $\overline{\Lambda}(\omega)$ are

(i) $$\overline{\Pi}(\omega) = \mathrm{sinc}\,(\omega),$$

(ii) $$\overline{\Lambda}(\omega) = \mathrm{sinc}^2\,(\omega).$$

Given Parseval's equality

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\overline{f}(\omega)|^2\, d\omega,$$

show that

(iii) $$\int_{-\infty}^{\infty} \mathrm{sinc}^2\,(\omega)\, d\omega = 2\pi,$$

(iv) $$\int_{-\infty}^{\infty} \mathrm{sinc}^4\,(\omega)\, d\omega = \frac{4\pi}{3}.$$

5. Given that $\overline{f}(s) = \mathcal{L}\{f(t)\}$ is the Laplace transform of $f(t)$, prove that when a is a constant

$$\mathcal{L}\{e^{at}f(t)\} \;=\; \overline{f}(s-a) \qquad\qquad s > a\,.$$

A 2nd order ordinary differential equation, with initial values, takes the form

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 20x = \delta(t-2)\,, \qquad x = \frac{dx}{dt} = 0 \text{ when } t = 0\,,$$

where δ represents the Dirac delta-function.

Use the Laplace convolution theorem to show that

$$x(t) = \begin{cases} \frac{1}{2}e^{-4(t-2)}\sin 2(t-2) & t > 2\,, \\ 0 & 0 \le t \le 2\,, \end{cases}$$

satisfies the differential equation and its initial conditions.

You may assume that $\mathcal{L}\{\sin\omega t\} = \dfrac{\omega}{s^2+\omega^2}$.

6. Two functions $f(t)$ and $g(t)$ have Laplace transforms $\overline{f}(s) = \mathcal{L}\{f(t)\}$ and $\overline{g}(s) = \mathcal{L}\{g(t)\}$ respectively.

If the convolution of $f(t)$ with $g(t)$ is defined as

$$f * g = \int_0^t f(u)g(t-u)\,du\,,$$

use double integration, with a change of order, to prove the Laplace convolution theorem

$$\mathcal{L}\{f * g\} = \overline{f}(s)\,\overline{g}(s).$$

Hence, or otherwise, show that

$$\mathcal{L}^{-1}\left\{\frac{1}{(1+s^2)^2}\right\} = \frac{1}{2}\left(\sin t - t\cos t\right)\,.$$

You may assume that $\mathcal{L}\{\sin\omega t\} = \dfrac{\omega}{s^2+\omega^2}$.

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7. (i) Consider a two-dimensional region R bounded by a closed piecewise smooth curve C. Green's Theorem in a plane states that for a two-dimensional region R bounded by a closed, piecewise smooth curve C,

$$\oint_C \{P(x,y)dx + Q(x,y)dy\} = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

where $P(x,y)$ and $Q(x,y)$ are arbitrary differentiable functions. Using this theorem, choose the components of a vector field $\boldsymbol{v}(x,y)$ in terms of $P(x,y)$ and $Q(x,y)$ to prove the two-dimensional form of Stokes' Theorem

$$\oint_C \boldsymbol{v}.d\boldsymbol{r} \;=\; \int\int_R \boldsymbol{k}.(\mathrm{curl}\,\boldsymbol{v})\,dxdy\,, \qquad \boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j}\,.$$

(ii) With a suitable choice of P and Q, show that

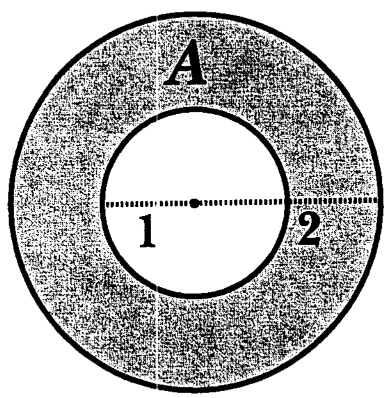
$$\oint_C (-ydx + xdy) = \int\int_R dxdy\,. \tag{1}$$

(iii) If R is the region whose upper and lower boundaries are the line $y = x$ and the curve $y = \frac{1}{2}x^2$ and whose left and right boundaries are the vertical lines $x = 1$ and $x = 2$, sketch the region R in the $x - y$ plane and, by evaluating the line integral on the right hand side of (1) and the double integral on the left hand side of (1), show that they both take the value 2/3.

8. Compute the Jacobian matrix $\mathcal{J}$ of the co-ordinate transformation from polar to Cartesian co-ordinates given by

$$x = r\cos\theta\,,$$
$$y = r\sin\theta\,.$$

Let A be the annulus between two concentric circles, centred on the origin, of radius 1 and 2 respectively as in the following figure



By transforming to polar co-ordinates and using the fact that $dx\,dy = |\det(\mathcal{J})|\,dr\,d\theta$ or otherwise, compute the following integrals

$$\iint_A dx\,dy\,,$$

$$\iint_A x\,dx\,dy\,,$$

$$\iint_A (x^2 + y^2)\,dx\,dy\,.$$

PLEASE TURN OVER

9. Let $\boldsymbol{F} = (F_1,\, F_2,\, F_3)$ be a vector field and φ a scalar field in three dimensions.

Define $\operatorname{grad}\varphi$, $\operatorname{div}\boldsymbol{F}$ and $\operatorname{curl}\boldsymbol{F}$.

(i) Suppose that F_1 depends only on x, F_2 depends only on y and F_3 depends only on z.

Show that

$$\operatorname{grad}(\boldsymbol{F}\,.\,\boldsymbol{F}) = 2\left(F_1\frac{\partial F_1}{\partial x},\ F_2\frac{\partial F_2}{\partial y},\ F_3\frac{\partial F_3}{\partial z}\right)$$

and calculate $\operatorname{grad}(\boldsymbol{F}\,.\,\boldsymbol{F})$ for the case

$$\boldsymbol{F} = (x^2,\, y^2,\, z^2)\ .$$

(ii) Let $\boldsymbol{v}$ be a *constant* vector. Calculate $\operatorname{div}(\boldsymbol{F}\times\boldsymbol{v})$, where $\boldsymbol{F}\times\boldsymbol{v}$ is the cross (vector) product of $\boldsymbol{F}$ and $\boldsymbol{v}$ and hence show that

$$\operatorname{div}(\boldsymbol{F}\times\boldsymbol{v}) = (\operatorname{curl}\boldsymbol{F})\,.\boldsymbol{v}\,.$$

(iii) Suppose that $F_3 = 0$ and F_1, F_2 depend only on x and y.

Show that $\operatorname{curl}\boldsymbol{F}$ is a vector in the z direction.

Calculate $\operatorname{curl}\boldsymbol{F}$ for the case

$$\boldsymbol{F} = (y^2,\, x^2,\, 0)\ .$$

10. (i) Show that the line integral

$$\int_C \left[x\,(\cos y\ +\ 1)\,dx\ -\ \frac{1}{2}\,x^2\,\sin y\,dy\right]$$

is independent of the path C joining the initial point of the path to the final point.

Evaluate the integral for a path C from $(0,\,0)$ to $(1,\,\pi/2)$.

(ii) Let R be the square region defined by $0 \le x \le 1$ and $0 \le y \le 1$.

Let C be the boundary of the region taken in the counter-clockwise direction.

Evaluate

$$\oint_C \left[-y^2\,dx\ +\ x^2\,dy\right],$$

(a) directly, and

(b) by using Green's theorem in the plane.

Green's theorem in the plane states that

$$\oint_C (P\,dx\ +\ Q\,dy)\ =\ \iint_R \left(\frac{\partial Q}{\partial x}\ -\ \frac{\partial P}{\partial y}\right)\,dx\,dy\ .$$

11. An electrical system consists of two subsystems, A and B, that operate independently. Each subsystem can be in one of three states, *down*, *dormant* or *active*. The probabilities of finding A in these states are $1/3$, $1/2$ and $1/6$, respectively, and the corresponding probabilities for subsystem B are $1/3$, $1/3$ and $1/3$. The system is *active* only when A and B are both *active*, and the system is *down* if either A or B is *down* (or both); otherwise, the system is *dormant*. Calculate the probabilities that the system is *down, dormant* or *active*.

Raising subsystem A from *down* to *active* incurs a cost of 2 units, and raising it from *dormant* to *active* costs 1 unit. The corresponding costs for subsystem B are 3 units and 1 unit. Compute the expected cost of achieving system state *active*, given that, initially, neither A nor B is *active*. What is this cost if it is known beforehand that subsystem B is *dormant*?

12. The random variables X_1 and X_2 have a joint probability distribution in which X_1 can take values 0 and 1, with respective probabilities $1/3$ and $2/3$, and X_2 can take values -1, 0 and 1. The conditional distribution of X_2 given X_1 can be summarised by

$$P(X_2 = -1|X_1 = 0) = \frac{1}{5}, \quad P(X_2 = 1|X_1 = 0) = \frac{2}{5},$$

$$P(X_2 = -1|X_1 = 1) = \frac{2}{5}, \quad P(X_2 = 1|X_1 = 1) = \frac{2}{5}.$$

(i) Draw up a table showing the joint probabilities $P(X_1 = x_1,\ X_2 = x_2)$ for all possible pairs $(x_1,\ x_2)$.

(ii) Calculate $P(X_2 \geq 0)$, $P(X_2 < X_1)$ and $P(X_1 = 0|X_2 \geq 0)$.

(iii) Find the ratio of means, $E(X_2)/E(X_1)$, and the ratio of variances, $\mathrm{var}(X_2)/\mathrm{var}(X_1)$.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\ a_2,\ a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\ \mathbf{b},\ \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary},\ |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\ ,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots\ ,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots\ ,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\ ;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\ g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!,\quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x,\ y)$ about $(a,\ b)$:

$$f(a+h,\ b+k) = f(a,\ b) + [hf_x + kf_y]_{a.b} + 1/2!\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a.b} + \ldots$$

(d) Partial differentiation of $f(x,\ y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t),\ y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u,\ v),\ y = y(u,\ v)$, then $f = F(u,\ v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x,\ y)$ occur where $f_x = 0,\ f_y = 0$ simultaneously.
Let $(a,\ b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D > 0$ and $f_{xx}(a,\ b) < 0$, then $(a,\ b)$ is a maximum;
If $D > 0$ and $f_{xx}(a,\ b) > 0$, then $(a,\ b)$ is a minimum;
If $D < 0$ then $(a,\ b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\ y)dx + Q(x,\ y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

1. Probabilities for events

For events A, B, and C $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally $\quad P(\bigcup A_i) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$

The odds in favour of A $\quad P(A) \,/\, P(\overline{A})$

Conditional probability $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$

Chain rule $\quad P(A \cap B \cap C) = P(A)\, P(B \mid A)\, P(C \mid A \cap B)$

Bayes' rule $\quad P(A \mid B) = \dfrac{P(A)\, P(B \mid A)}{P(A)\, P(B \mid A) + P(\overline{A})\, P(B \mid \overline{A})}$

A and B are independent if $\quad P(B \mid A) = P(B)$

A, B, and C are independent if $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$, and

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(C \cap A) = P(C)P(A)$$

2. Probability distribution, expectation and variance

The probability distribution for a discrete random variable X is the complete set of probabilities $\{p_x\} = \{P(X = x)\}$

Expectation $\quad E(X) = \mu = \sum_x x p_x$

Sample mean $\quad \overline{x} = \dfrac{1}{n} \sum_k x_k$ estimates μ from random sample $x_1, x_2, \ldots, x_n$

Variance $\quad \operatorname{var}(X) = \sigma^2 = E\{(X - \mu)^2\} = E(X^2) - \mu^2$, where $E(X^2) = \sum_x x^2 p_x$

Sample variance $\quad s^2 = \dfrac{1}{n-1} \left\{ \sum_k x_k^2 - \dfrac{1}{n} \left(\sum_j x_j \right)^2 \right\}$ estimates σ^2

Standard deviation $\quad \operatorname{sd}(X) = \sigma$

If value y is observed with frequency n_y

$$n = \sum_y n_y, \quad \sum_k x_k = \sum_y y n_y, \quad \sum_k x_k^2 = \sum_y y^2 n_y$$

For function $g(x)$ of x, $\quad E\{g(X)\} = \sum_x g(x) p_x$

Skewness $\quad \beta_1 = E\left(\dfrac{X - \mu}{\sigma}\right)^3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^3$

Kurtosis $\quad \beta_2 = E\left(\dfrac{X - \mu}{\sigma}\right)^4 - 3$ is estimated by $\dfrac{1}{n-1} \sum \left(\dfrac{x_i - \overline{x}}{s}\right)^4 - 3$

Sample median $\tilde{x}$. If the sample values $x_1, \ldots, x_n$ are ordered $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$

$\tilde{x} = x_{(\frac{n+1}{2})}$ if n is odd, and $\tilde{x} = \frac{1}{2}(x_{(\frac{n}{2})} + x_{(\frac{n+2}{2})})$ if n is even.

α-quantile $Q(\alpha)$ is such that $P(X \leq Q(\alpha)) = \alpha$

Sample α-quantile $\hat{Q}(\alpha)$ is the sample value for which the proportion of values $\leq \hat{Q}(\alpha)$ is α (using linear interpolation between values on either side)

The sample median $\tilde{x}$ estimates the population median $Q(0.5)$.

3. Probability distribution for a continuous random variable

The cumulative distribution function (cdf) $F(x) = P(X \leq x) = \int_{x_0=-\infty}^{x} f(x_0)\mathrm{d}x_0$

The probability density function (pdf) $f(x) = \dfrac{\mathrm{d}F(x)}{\mathrm{d}x}$

$$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x, \quad \mathrm{var}\,(X) = \sigma^2 = E(X^2) - \mu^2,$$

where $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\mathrm{d}x$

4. Discrete probability distributions

Discrete Uniform *Uniform* (n)

$p_x = \dfrac{1}{n} \quad (x = 1, 2, \ldots, n)$ $\qquad \mu = \frac{1}{2}(n+1), \ \sigma^2 = \frac{1}{12}(n^2 - 1)$

Binomial distribution *Binomial* (n, θ)

$p_x = \dbinom{n}{x} \theta^x (1-\theta)^{n-x} \quad (x = 0, 1, 2, \ldots, n)$ $\qquad \mu = n\theta, \ \sigma^2 = n\theta(1-\theta)$

Poisson distribution *Poisson* (λ)

$p_x = \dfrac{\lambda^x e^{-\lambda}}{x!} \quad (x = 0, 1, 2, \ldots)$ (with $\lambda > 0$) $\qquad \mu = \lambda, \ \sigma^2 = \lambda$

Geometric distribution *Geometric* (θ)

$p_x = (1-\theta)^{x-1}\theta \quad (x = 1, 2, 3, \ldots)$ $\qquad \mu = \dfrac{1}{\theta}, \ \sigma^2 = \dfrac{1-\theta}{\theta^2}$

5. Continuous probability distributions

Uniform distribution *Uniform* (α, β)

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & (\alpha < x < \beta), \\ 0 & (\text{otherwise}). \end{cases}$$

$\mu = (\alpha + \beta)/2,$

$\sigma^2 = (\beta - \alpha)^2/12.$

Exponential distribution $Exponential\,(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (0 < x < \infty), \\ 0 & (-\infty < x \le 0). \end{cases} \qquad \mu = 1/\lambda, \quad \sigma^2 = 1/\lambda^2.$$

Normal distribution $N\,(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} \quad (-\infty < x < \infty)$$

$$E(X) = \mu, \quad \text{var}\,(X) = \sigma^2$$

Standard normal distribution $N\,(0,1)$

If X is $N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma}$ is $N(0,1)$

6. Reliability

For a device in continuous operation with failure time random variable T having pdf $f(t)$ $(t > 0)$

The reliability function at time t	$R(t) = P(T > t)$
The failure rate or hazard function	$h(t) = f(t)/R(t)$
The cumulative hazard	$H(t) = \int_0^t h(t_0)\,dt_0 = -\ln\{R(t)\}$
The Weibull(α, β) distribution has	$H(t) = \beta t^\alpha$

7. System reliability

For a system of k devices, which operate independently, let

$R_i = P(D_i) = P($"device i operates"$)$

The system reliability, R, is the probability of a path of operating devices

A system of devices in series operates only if every device operates

$$R = P(D_1 \cap D_2 \cap \cdots \cap D_k) = R_1 R_2 \cdots R_k$$

A system of devices in parallel operates if any device operates

$$R = P(D_1 \cup D_2 \cup \cdots \cup D_k) = 1 - (1-R_1)(1-R_2)\cdots(1-R_k)$$

8. Covariance and correlation

The covariance of X and Y $\quad \text{cov}\,(X,Y) = E(XY) - \{E(X)\}\{E(Y)\}$

From pairs of observations $(x_1, y_1), \ldots, (x_n, y_n)$ $\quad S_{xy} = \sum_k x_k y_k - \frac{1}{n}(\sum_i x_i)(\sum_j y_j)$

$$S_{xx} = \sum_k x_k^2 - \frac{1}{n}(\sum_i x_i)^2, \qquad S_{yy} = \sum_k y_k^2 - \frac{1}{n}(\sum_j y_j)^2$$

Sample covariance $\quad s_{xy} = \frac{1}{n-1} S_{xy}$ estimates $\operatorname{cov}(X, Y)$

Correlation coefficient $\quad \rho = \operatorname{corr}(X, Y) = \frac{\operatorname{cov}(X, Y)}{\operatorname{sd}(X) \cdot \operatorname{sd}(Y)}$

Sample correlation coefficient $\quad r = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$ estimates ρ

9. Sums of random variables

$$E(X+Y) = E(X) + E(Y)$$

$$\operatorname{var}(X+Y) = \operatorname{var}(X) + \operatorname{var}(Y) + 2\operatorname{cov}(X, Y)$$

$$\operatorname{cov}(aX + bY,\ cX + dY) = (ac)\operatorname{var}(X) + (bd)\operatorname{var}(Y) + (ad + bc)\operatorname{cov}(X, Y)$$

If X is $N(\mu_1, \sigma_1^2)$, Y is $N(\mu_2, \sigma_2^2)$, and $\operatorname{cov}(X, Y) = c$,
then $X + Y$ is $N(\mu_1 + \mu_2,\ \sigma_1^2 + \sigma_2^2 + 2c)$

10. Bias, standard error, mean square error

If t estimates θ (with random variable T giving t)

Bias of t $\quad \text{bias}(t) = E(T) - \theta$

Standard error of t $\quad \operatorname{se}(t) = \operatorname{sd}(T)$

Mean square error of t $\quad \text{MSE}(t) = E\{(T-\theta)^2\} = \{\operatorname{se}(t)\}^2 + \{\text{bias}(t)\}^2$

If $\bar{x}$ estimates μ, then $\text{bias}(\bar{x}) = 0$, $\operatorname{se}(\bar{x}) = \sigma/\sqrt{n}$, $\text{MSE}(\bar{x}) = \sigma^2/n$, $\widehat{\operatorname{se}}(\bar{x}) = s/\sqrt{n}$

Central limit property $\quad$ if n is fairly large, $\bar{x}$ is from $N(\mu,\ \sigma^2/n)$ approximately

11. Likelihood

The likelihood is the joint probability as a function of the unknown parameter θ.
For a random sample $x_1, x_2, \ldots, x_n$

$$\ell(\theta;\ x_1, x_2, \ldots, x_n) = P(X_1 = x_1 \mid \theta) \cdots P(X_n = x_n \mid \theta) \quad \text{(discrete distribution)}$$

$$\ell(\theta;\ x_1, x_2, \ldots, x_n) = f(x_1 \mid \theta) f(x_2 \mid \theta) \cdots f(x_n \mid \theta) \quad \text{(continuous distribution)}$$

The maximum likelihood estimator (MLE) is $\hat{\theta}$ for which the likelihood is a maximum.

12. Confidence intervals

If $x_1, x_2, \ldots, x_n$ are a random sample from $N(\mu, \sigma^2)$ and σ^2 is known, then the 95% confidence interval for μ is $(\overline{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \overline{x} + 1.96\frac{\sigma}{\sqrt{n}})$

If σ^2 is estimated, then from the Student t table for t_{n-1} we find $t_0 = t_{n-1,0.05}$

The 95% confidence interval for μ is $(\overline{x} - t_0\frac{s}{\sqrt{n}},\ \overline{x} + t_0\frac{s}{\sqrt{n}})$

13. Standard normal table

Values of pdf $\phi(y) = f(y)$ and cdf $\Phi(y) = F(y)$

y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\phi(y)$	$\Phi(y)$	y	$\Phi(y)$
0	.399	.5	.9	.266	.816	1.8	.079	.964	2.8	.997
.1	.397	.540	1.0	.242	.841	1.9	.066	.971	3.0	.998
.2	.391	.579	1.1	.218	.864	2.0	.054	.977	0.841	.8
.3	.381	.618	1.2	.194	.885	2.1	.044	.982	1.282	.9
.4	.368	.655	1.3	.171	.903	2.2	.035	.986	1.645	.95
.5	.352	.691	1.4	.150	.919	2.3	.028	.989	1.96	.975
.6	.333	.726	1.5	.130	.933	2.4	.022	.992	2.326	.99
.7	.312	.758	1.6	.111	.945	2.5	.018	.994	2.576	.995
.8	.290	.788	1.7	.094	.955	2.6	.014	.995	3.09	.999

14. Student t table

Values $t_{m,p}$ of x for which $P(|X| > x) = p$, when X is t_m

	p	.10	.05	.02	0.01		p	.10	.05	.02	0.01
m	1	6.31	12.71	31.82	63.66	m	9	1.83	2.26	2.82	3.25
	2	2.92	4.30	6.96	9.92		10	1.81	2.23	2.76	3.17
	3	2.35	3.18	4.54	5.84		12	1.78	2.18	2.68	3.05
	4	2.13	2.78	3.75	4.60		15	1.75	2.13	2.60	2.95
	5	2.02	2.57	3.36	4.03		20	1.72	2.09	2.53	2.85
	6	1.94	2.45	3.14	3.71		25	1.71	2.06	2.48	2.78
	7	1.89	2.36	3.00	3.50		40	1.68	2.02	2.42	2.70
	8	1.86	2.31	2.90	3.36		∞	1.645	1.96	2.326	2.576

15. Chi-squared table

Values $\chi^2_{k,p}$ of x for which $P(X > x) = p$, when X is χ^2_k and $p = .995,\ .975,\ etc$

k	.995	.975	.05	.025	.01	.005	k	.995	.975	.05	.025	.01	.005
1	.000	.001	3.84	5.02	6.63	7.88	18	6.26	8.23	28.87	31.53	34.81	37.16
2	.010	.051	5.99	7.38	9.21	10.60	20	7.43	9.59	31.42	34.17	37.57	40.00
3	.072	.216	7.81	9.35	11.34	12.84	22	8.64	10.98	33.92	36.78	40.29	42.80
4	.207	.484	9.49	11.14	13.28	14.86	24	9.89	12.40	36.42	39.36	42.98	45.56
5	.412	.831	11.07	12.83	15.09	16.75	26	11.16	13.84	38.89	41.92	45.64	48.29
6	.676	1.24	12.59	14.45	16.81	18.55	28	12.46	15.31	41.34	44.46	48.28	50.99
7	.990	1.69	14.07	16.01	18.48	20.28	30	13.79	16.79	43.77	46.98	50.89	53.67
8	1.34	2.18	15.51	17.53	20.09	21.95	40	20.71	24.43	55.76	59.34	63.69	66.77
9	1.73	2.70	16.92	19.02	21.67	23.59	50	27.99	32.36	67.50	71.41	76.15	79.49
10	2.16	3.25	13.31	20.48	23.21	25.19	60	35.53	40.48	79.08	83.30	88.38	91.95
12	3.07	4.40	21.03	23.34	26.22	28.30	70	43.28	48.76	90.53	95.02	100.4	104.2
14	4.07	5.63	23.68	26.12	29.14	31.32	80	51.17	57.15	101.9	106.6	112.3	116.3
16	5.14	6.91	26.30	28.85	32.00	34.27	100	67.33	74.22	124.3	129.6	135.8	140.2

16. The chi-squared goodness-of-fit test

The frequencies n_y are grouped so that the fitted frequency $\widehat{n}_y$ for every group exceeds about 5.

$X^2 = \sum_y \frac{(n_y - \widehat{n}_y)^2}{\widehat{n}_y}$ is referred to the table of χ^2_k with significance point p,

where k is the number of terms summed, less one for each constraint, *eg* matching total frequency, and matching $\overline{x}$ with μ.

17. Joint probability distributions

Discrete distribution $\{p_{xy}\}$, where $p_{xy} = P(\{X = x\} \cap \{Y = y\})$.

Let $p_{x\bullet} = P(X = x)$, and $p_{\bullet y} = P(Y = y)$, then

$p_{x\bullet} = \sum_y p_{xy}$, and $P(X = x \mid Y = y) = \frac{p_{xy}}{p_{\bullet y}}$.

Continuous distribution

Joint cdf $F(x,y) = P(\{X \le x\} \cap \{Y \le y\}) = \int_{x_0=-\infty}^{x} \int_{y_0=-\infty}^{y} f(x_0, y_0)\, dx_0\, dy_0$

Joint pdf $f(x,y) = \dfrac{d^2F(x,y)}{dx\,dy}$

Marginal pdf of X $f_X(x) = \int_{-\infty}^{\infty} f(x, y_0)\, dy_0$

Conditional pdf of X given $Y = y$ $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ (provided $f_Y(y) > 0$)

18. **Linear regression**

To fit the linear regression model $y = \alpha + \beta x$ by $\hat{y}_x = \hat{\alpha} + \hat{\beta} x$ from observations $(x_1, y_1), \ldots, (x_n, y_n)$, the least squares fit is

$$\hat{\alpha} = \overline{y} - \overline{x}\hat{\beta}, \quad \hat{\beta} = S_{xy}/S_{xx}$$

The residual sum of squares RSS $= S_{yy} - \dfrac{S_{xy}^2}{S_{xx}}$

$$\widehat{\sigma^2} = \frac{\text{RSS}}{n-2}, \qquad \frac{n-2}{\sigma^2}\widehat{\sigma^2} \text{ is from } \chi^2_{n-2}$$

$$E(\hat{\alpha}) = \alpha, \quad E(\hat{\beta}) = \beta,$$

$$\text{var}(\hat{\alpha}) = \frac{\sum x_i^2}{n S_{xx}}\sigma^2, \quad \text{var}(\hat{\beta}) = \frac{\sigma^2}{S_{xx}}, \quad \text{cov}(\hat{\alpha}, \hat{\beta}) = -\frac{\overline{x}}{S_{xx}}\sigma^2$$

$$\hat{y}_x = \hat{\alpha} + \hat{\beta} x, \qquad E(\hat{y}_x) = \alpha + \beta x, \qquad \text{var}(\hat{y}_x) = \left\{\frac{1}{n} + \frac{(x - \overline{x})^2}{S_{xx}}\right\}\sigma^2$$

$$\frac{\hat{\alpha} - \alpha}{\widehat{\text{se}}(\hat{\alpha})}, \quad \frac{\hat{\beta} - \beta}{\widehat{\text{se}}(\hat{\beta})}, \quad \frac{\hat{y}_x - \alpha - \beta x}{\widehat{\text{se}}(\hat{y}_x)} \text{ are each from } t_{n-2}$$

19. **Design matrix for factorial experiments** With 3 factors each at 2 levels

$$\mathbf{X} = \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2004 – 2005

E1

PAPER 3

QUESTION

SOLUTION A5

Please write on this side only, legibly and neatly, between the margins

$$w = \frac{1}{z-4}$$

a) $u + iv = \dfrac{1}{x-4+iy} = \dfrac{x-4-iy}{(x-4)^2+y^2}$ 3

$u = \dfrac{x-4}{(x-4)^2+y^2}$; $v = -\dfrac{y}{(x-4)^2+y^2}$ — (*)

b) $u^2+v^2 = \dfrac{1}{(x-4)^2+y^2} = (1/5)^2$ Circle centred at (0,0) radius 1/5. 2

c) In (*) $y = x-4$ gives $u = \dfrac{1}{2y}$, $v = -\dfrac{1}{2y}$ 3

$\therefore$ $v = -u$ in the w-plane.

d) In (*) $x = 0$ gives

$u = \dfrac{-4}{y^2+16}$; $v = -\dfrac{y}{y^2+16}$

$\therefore$ $u^2+v^2 = \left[\dfrac{1}{(x-4)^2+y^2}\right]_{x=0} = \dfrac{1}{y^2+16} = -u/4$ 3

$\therefore$ $(u+1/8)^2 + v^2 = (1/8)^2$ completing the square.

Circle centred at $(-1/8, 0)$ radius 1/8 in w-plane.

e) In (*) $x = 4$ gives $u = 0$, $v = -1/y$. Thus 2

$x = 4$ maps to the v-axis ($u = 0$) in the w-plane.

f) Fixed points z_0 satisfy $z_0 = w_0 = \dfrac{1}{z_0-4}$

$\therefore$ $z_0^2 - 4z_0 - 1 = 0$ 2

$z_0 = \frac{1}{2}\left[4 \pm \{16+4\}^{1/2}\right]$

$= [2 \pm \sqrt{5}]$ both on real axis.

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Checker : R.L. JACOBS Checker's signature : R.L. Jacobs

15

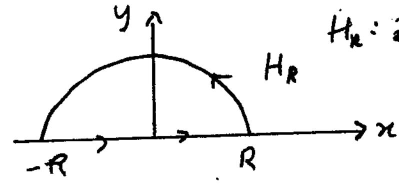
PAPER 3

QUESTION

SOLUTION E2

$$\oint_C e^{iz} F(z)\,dz$$

$F(z)$ has poles as its only singularities



$H_R : z = Re^{i\theta}$

$$\oint_C e^{iz} F(z)\,dz = 2\pi i \times \{\text{Sum of residues of } e^{iz}F(z) \text{ at its poles in upper } \tfrac{1}{2}\text{-plane}\}$$ 5

$$F(z) = \frac{1}{(z^2+4)^2}$$

$$= \int_{-\infty}^{\infty} e^{ix} F(x)\,dx + \lim_{R\to\infty} \int_{H_R} e^{iz} F(z)\,dz$$

$e^{iz}F(z)$ has a double pole at $z = 2i$

Residue: $$\lim_{z\to 2i} \left\{ \frac{d}{dz} \frac{(z-2i)^2 e^{iz}}{(z^2+4)^2} \right\}$$

$$= \lim_{z\to 2i} \frac{d}{dz}\left[\frac{e^{iz}}{(z+2i)^2}\right]$$ 5

$$= \left\{ e^{iz}\left[\frac{i(z+2i)-2}{(z+2i)^3}\right]\right\}_{z=2i} = e^{-2}\left\{\frac{-4-2}{(4i)^3}\right\}$$

$$= 3e^{-2}/32i$$

$$\therefore \frac{3\pi e^{-2}}{16} = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)^2}\,dx + \underbrace{\lim_{R\to\infty}\int_{H_R} \frac{e^{iz}}{(z^2+4)^2}\,dz}$$

zero by Jordan's Lemma
i) Only poles
ii) $m = 1 > 0$
iii) $|F(z)| \to 0$ as $R \to \infty$. 1 + 3

Moreover $$\int_{-\infty}^{\infty} \frac{\sin x\,dx}{(x^2+4)^2} = 0$$ by symmetry

$$\therefore \int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+4)^2} = \frac{3\pi}{16e^2}$$ 1

(15)

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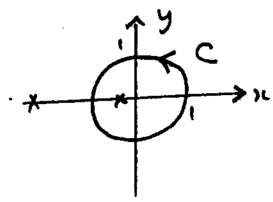
Checker : Y. Chen Checker's signature : Chen

$$I = \int_0^{2\pi} \frac{d\theta}{3+\cos\theta} \qquad \cos\theta = \tfrac{1}{2}(e^{i\theta}+e^{-i\theta}) = \tfrac{1}{2}(z + 1/z)$$

$$dz = ie^{i\theta}\,d\theta = iz\,d\theta$$

$$I = \frac{2}{i}\oint_C \frac{z^{-1}dz}{6+(z+z^{-1})} = \frac{2}{i}\oint_C \frac{dz}{z^2+6z+1}$$ 5

where C is the unit circle



Now $z^2+6z+1 = (z+3)^2-8$ 5

∴ There are 2 simple poles on the real axis at

$z = -3 \pm \sqrt{8}$ + inside C; − outside C ← ignore

$$\therefore\ I = 2\pi i \times \left\{ \frac{2}{i} \times \text{Residue of } (z^2+6z+1)^{-1} \text{ at the pole at } z_+ = -3+\sqrt{8} \right\}$$ 2

Residue is $\lim_{z\to z_+}\left\{\frac{(z-z_+)}{(z-z_+)(z-z_-)}\right\} = \frac{1}{z_+-z_-}$ 3

$$= \frac{1}{2\sqrt{8}}$$

$$\therefore\ I = \frac{2\pi}{\sqrt{8}}$$

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(15)

i) $\bar{\Pi}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\,\Pi(t)\,dt = \int_{-1/2}^{1/2} 1 \cdot e^{-i\omega t}\,dt$

$= -\frac{1}{i\omega}\left[e^{-i\omega t}\right]_{-1/2}^{1/2} = \frac{2\left(e^{i\omega/2} - e^{-i\omega/2}\right)}{2i\omega}$

$= \frac{\sin \omega/2}{\omega/2} = \operatorname{sinc} \omega$ 4

ii) $\bar{\Lambda}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\Lambda(t)\,dt = \int_{-1}^{0}(1+t)e^{-i\omega t}dt + \int_{0}^{1}(1-t)e^{-i\omega t}dt$

$= \int_{-1}^{1} e^{-i\omega t}dt + \underbrace{\int_{-1}^{0} t\,e^{-i\omega t}dt}_{t \to -t} - \int_0^1 t\,e^{-i\omega t}dt$

$= -\int_0^1 t\,e^{i\omega t}dt$

$\therefore \bar{\Lambda}(\omega) = \int_{-1}^{1} e^{-i\omega t}dt - 2\int_0^1 t\cos\omega t\,dt$ 5

$\int_0^1 t\cos\omega t\,dt = \frac{1}{\omega}\int_0^1 t\,d(\sin\omega t) = \frac{\sin\omega}{\omega} + \frac{\cos\omega - 1}{\omega^2}$

$\therefore \bar{\Lambda}(\omega) = \frac{2\sin\omega}{\omega} - 2\left[\frac{\sin\omega}{\omega} + \frac{\cos\omega - 1}{\omega^2}\right] = \frac{4\sin^2\frac{1}{2}\omega}{\omega^2}$

iii) $\int_{-\infty}^{\infty} \operatorname{sinc}^2\omega\,d\omega = 2\pi\int_{-\infty}^{\infty}|\Pi(t)|^2\,dt = 2\pi$ $\quad = \operatorname{sinc}^2 w.$ 2

iv) $\int_{-\infty}^{\infty} \operatorname{sinc}^4\omega\,d\omega = 2\pi\int_{-\infty}^{\infty}|\Lambda(t)|^2\,dt$

$= 2\pi\left\{\int_{-1}^{0}(1+t)^2dt + \int_0^1(1-t)^2dt\right\}$

$= 4\pi\int_0^1(1-t)^2\,dt$ as $\int_{-1}^{0}(1+t)^2dt$ $t \to -t$ $= \int_0^1(1-t)^2dt$

$= 4\pi\left[t - t^2 + \frac{1}{3}t^3\right]_0^1$ 4

$= 4\pi/3$

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Checker : Y. C.   Checker's signature : Cheng

(15)

PAPER 3

QUESTION

SOLUTION E5

See below for bookwork!

$$\ddot{x} + 8\dot{x} + 20x = \delta(t-2) \qquad x(0) = 0, \quad \dot{x}(0) = 0$$

$$\therefore \mathcal{L}(\dot{x}) = s\bar{x}(s)$$

$$\mathcal{L}(\ddot{x}) = s^2\bar{x}(s)$$

$$\therefore (s^2 + 8s + 20)\bar{x}(s) = \int_0^\infty e^{-st}\delta(t-2)\,dt = e^{-2s} \quad s>0$$

$$\therefore \bar{x}(s) = \frac{e^{-2s}}{s^2+8s+20} = \frac{e^{-2s}}{(s+4)^2+4}$$ 3

Choose $\bar{f}(s) = e^{-2s}$

$$\bar{g}(s) = \frac{1}{(s+4)^2+4}$$ 3

$$\therefore f(t) = \delta(t-2)$$

$$g(t) = \tfrac{1}{2}e^{-4t}\sin 2t$$ by shift theorem & inverse.

$$\therefore x(t) = \mathcal{L}^{-1}[\bar{f}(s)\bar{g}(s)]$$ 3

$$= f(t) * g(t)$$

$$= \tfrac{1}{2}\int_0^t \delta(u-2)\, e^{-4(t-u)}\sin 2(t-u)\,du$$ 3

$$= \begin{cases} \tfrac{1}{2}e^{-4(t-2)}\sin 2(t-2) & t>2 \\ 0 & 0 \le t \le 2 \end{cases}$$

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-(s-a)t}f(t)\,dt = \bar{f}(s-a)$$ 3

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Checker : Y.C.

Checker's signature : [signature]

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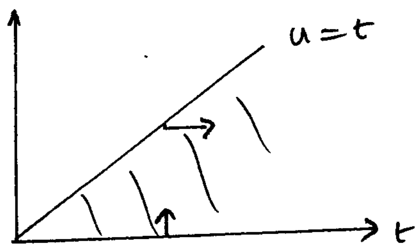
$$\mathcal{L}(f*g) = \int_0^\infty e^{-st}\left\{\int_0^t f(u)g(t-u)\,du\right\}dt$$

change the order of integn. 2



$$= \int_0^\infty f(u)\left(\int_u^\infty e^{-st} g(t-u)\,dt\right)du$$ 3 (pic)

Let $\tau = t-u$

$$= \int_0^\infty f(u)\left(\int_0^\infty e^{-s(\tau+u)} g(\tau)\,d\tau\right)du$$ 2

$$= \int_0^\infty e^{-su} f(u)\,du \int_0^\infty e^{-s\tau} g(\tau)\,d\tau = \bar{f}(s)\bar{g}(s)$$

$$\mathcal{L}(f*f) = [\bar{f}(s)]^2 \equiv \frac{1}{(1+s^2)^2}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(1+s^2)^2}\right\} = f*f \quad \text{where } \bar{f}(s) = \frac{1}{1+s^2}$$ 4

$$\Rightarrow f(t) = \sin t$$

$$\therefore f*f = \int_0^t \sin u \sin(t-u)\,du$$

$$= \tfrac{1}{2}\int_0^t [\cos(2u-t) - \cos t]\,du$$

$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$
$A = u \qquad B = t-u$

$$= \tfrac{1}{4}[\sin(2u-t)]_0^t - \tfrac{1}{2}t\cos t$$ 4

$$= \tfrac{1}{4}[\sin t - \sin(-t)] - \tfrac{1}{2}t\cos t$$

$$= \tfrac{1}{2}[\sin t - t\cos t]$$

Note: an acceptable method is to use a parameter ω

$$\frac{\partial}{\partial\omega}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{1}{s^2+\omega^2} - \frac{2\omega^2}{(s^2+\omega^2)^2}$$ & invert

Setter: J.D. GIBBON — Setter's signature: J.D. Gibbon

Checker: Y... — Checker's signature: [signature]

15

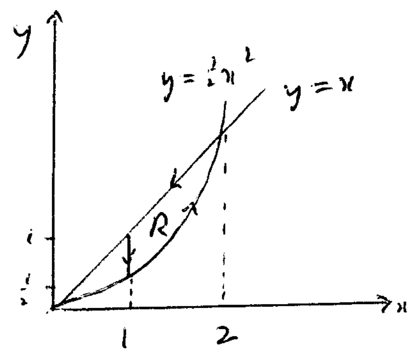
PAPER 3

QUESTION

SOLUTION E7

In G.T. choose $P = -y$, $Q = x$, so $\oint_C -y\,dx + x\,dy = 2\iint_R dx\,dy$ 1

$\oint_C \underline{v}\cdot d\underline{r} = \oint_C -y\,dx + x\,dy$



$\int_{y=\frac{1}{2}x^2} = \int_1^2 -\frac{1}{2}x^2\,dx + x^2\,dx$

$= \frac{1}{2}\int_1^2 x^2\,dx = \frac{1}{6}\left[x^3\right]_1^2 = 7/6.$ 1 (pic)

$\int_{y=x} = \int_2^1 (-x\,dx + x\,dx) = 0$

$\int_{x=1} = \int_{x=1} x\,dy = \int_1^{1/2} dy = -1/2$ Total: $7/6 - \frac{1}{2} = 2/3.$ 4

$\iint_R \underline{k}\cdot \text{curl}\,\underline{v}\,dx\,dy = 2\iint_R dx\,dy = 2\int_1^2 \left[y\right]_{\frac{1}{2}x^2}^{x} dx$

$= 2\int_1^2 \left(x - \frac{1}{2}x^2\right) dx = 2\left[\frac{1}{2}x^2 - \frac{1}{6}x^3\right]_1^2$

$= 4 - 8/3 - 1 + \frac{1}{3} = 3 - 7/3 = 2/3.$ 4

Bookwork part: Choose $\underline{v} = \underline{i}P + \underline{j}Q$. Because $\underline{r} = (x,y)$ we have $\underline{v}\cdot d\underline{r} = P\,dx + Q\,dy$. Moreover

$$\text{curl}\,\underline{v} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = \underline{k}\,(Q_x - P_y)$$ 5

∴ G.T. translates as $\oint_C \underline{v}\cdot d\underline{r} = \iint_R \hat{\underline{k}}\cdot \text{curl}\,\underline{v}\,dx\,dy$

Setter: J.D.G.   Setter's signature: J.D. Gibbon

Checker: Y.C.   Checker's signature: [signature]

(15)

C3 The Jacobian matrix is given by

$$\mathcal{J} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

2 Marks
Only the 2nd matrix is required

Thus

$$\det \mathcal{J} = r\cos^2\theta + r\sin^2\theta = r$$

1 Mark

and hence

$$dx\,dy = r\,dr\,d\theta$$

1 Mark

Thus, the first integral is

$$\iint_A dx\,dy = \iint_A r\,dr\,d\theta = \int_0^{2\pi}\left(\int_1^2 r\,dr\right)d\theta$$
$$= \int_0^{2\pi}\left[\tfrac{1}{2}r^2\right]_1^2 d\theta = \int_0^{2\pi}\tfrac{3}{2}\,d\theta$$
$$= 3\pi$$

3 Marks
Observing that the integral is the area of A and saying this is the difference in the area of the circles of radius 1 and 2 is acceptable

The second integral

$$\iint_A x\,dx\,dy = \iint_A r^2\cos\theta\,dr\,d\theta = \int_0^{2\pi}\cos\theta\left(\int_1^2 r^2\,dr\right)d\theta$$
$$= \int_0^{2\pi}\cos\theta\left[\tfrac{1}{3}r^3\right]_1^2 d\theta = \int_0^{2\pi}\tfrac{7}{3}\cos\theta\,d\theta$$
$$= \left[\sin\theta\right]_0^{2\pi} = 0$$

4 Marks
Arguing that by symmetry the answer must be 0 is acceptable

The third integral

$$\iint_A (x^2+y^2)\,dx\,dy = \iint_A r^3\,dr\,d\theta = \int_0^{2\pi}\left(\int_1^2 r^3\,dr\right)d\theta$$
$$= \int_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_1^2 d\theta = \int_0^{2\pi}\tfrac{15}{4}\,d\theta$$
$$= \tfrac{15}{2}\pi$$

4 Marks

J. Stark 9/2/05 (15)

 E9

C4 ~~The Jacobian matrix is given by~~

$$\text{grad } \varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$$

$$\text{div } \boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{curl } \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the x, y and z directions respectively.

3 Marks

Only one or the other form of curl needs to be given

a) We have

$$\boldsymbol{F}.\boldsymbol{F} = (F_1)^2 + (F_2)^2 + (F_3)^2$$

Since F_2 and F_3 are independent of x, we have

$$\frac{\partial}{\partial x}(\boldsymbol{F}.\boldsymbol{F}) = \frac{\partial}{\partial x}(F_1)^2 = 2F_1\frac{\partial F_1}{\partial x}$$

Similarly

$$\frac{\partial}{\partial y}(\boldsymbol{F}.\boldsymbol{F}) = \frac{\partial}{\partial y}(F_2)^2 = 2F_2\frac{\partial F_2}{\partial y}$$

$$\frac{\partial}{\partial z}(\boldsymbol{F}.\boldsymbol{F}) = \frac{\partial}{\partial z}(F_3)^2 = 2F_3\frac{\partial F_3}{\partial z}$$

and hence

$$\text{grad}(\boldsymbol{F}.\boldsymbol{F}) = \left(2F_1\frac{\partial F_1}{\partial x}, 2F_2\frac{\partial F_2}{\partial y}, 2F_3\frac{\partial F_3}{\partial z}\right)$$

3 Marks

Now if $\boldsymbol{F} = (x^2, y^2, z^2)$ we have

$$\text{grad}(\boldsymbol{F}.\boldsymbol{F}) = (2x^2 2x, 2y^2 2y, 2z^2 2z) = 4(x^3, y^3, 4z^3)$$

1 Mark

b) We have

$$\boldsymbol{F}\times\boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ F_1 & F_2 & F_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = ((F_2v_3 - F_3v_2), (F_3v_1 - F_1v_3), (F_1v_2 - F_2v_1))$$

so that

$$\text{div}(\boldsymbol{F}\times\boldsymbol{v}) = \frac{\partial}{\partial x}(F_2v_3 - F_3v_2) + \frac{\partial}{\partial y}(F_3v_1 - F_1v_3) + \frac{\partial}{\partial z}(F_1v_2 - F_2v_1))$$

4S

C.4

E9

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)v_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)v_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)v_3$$

2 Marks

Conversely

$$(\text{curl } \boldsymbol{F}).\boldsymbol{v} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right).(v_1, v_2, v_3)$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)v_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)v_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)v_3$$

Hence, as required

$$\text{div}(\boldsymbol{F} \times \boldsymbol{v}) = (\text{curl } \boldsymbol{F}).\boldsymbol{v}$$

2 Marks

c) We have

$$\text{curl } \boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

If $F_3 = 0$ and F_1 and F_2 are independent of z, we have

$$\text{curl } \boldsymbol{F} = (0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})$$

2 Marks

If $\boldsymbol{F} = (y^2, x^2, 0)$ we have

$$\text{curl } \boldsymbol{F} = (0, 0, 2x - 2y)$$

2 Marks

15

AS.

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2004 – 2005

E10

PAPER 3

QUESTION

SOLUTION C5

Please write on this side only, legibly and neatly, between the margins

(a) Line integral is independent of path if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, $\left.\begin{array}{l} Q = -\frac{1}{2}x^2 \sin y \\ P = x(\cos y + 1) \end{array}\right\}$ in this case

$\frac{\partial Q}{\partial x} = -x \sin y$, $\frac{\partial P}{\partial y} = -x \sin y$

Equal so independent of path. 3

Find a potential V such that

$\frac{\partial V}{\partial x} = P = x(\cos y + 1)$, $\frac{\partial V}{\partial y} = Q = -\frac{1}{2}x^2 \sin y$

From $\frac{\partial V}{\partial x}$, $V = \frac{x^2}{2}(\cos y + 1) + f(y)$

Subst. into $\frac{\partial V}{\partial y}$ to get $-\frac{1}{2}x^2 \sin y + \frac{df}{dy} = -\frac{1}{2}x^2 \sin y$ 3

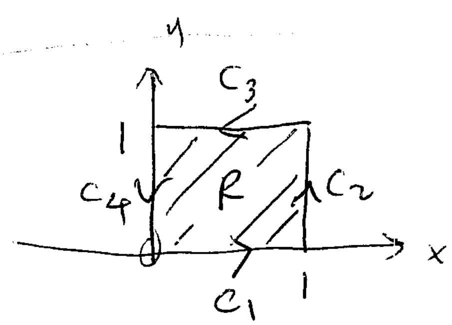
$\Rightarrow f(y) = c$

Thus integral from $(0,0)$ to $(1, \frac{\pi}{2})$

is $V(1, \frac{\pi}{2}) - V(0,0) = \left[\frac{1}{2}(0+1) + c\right] - [0 + c]$ 2

$= \frac{1}{2}$ 3

(b)



On C_1 $y = 0$ $dy = 0$

On C_2 $x = 1$ $dx = 0$

On C_3 $y = 1$ $dy = 0$

On C_4 $x = 0$ $dx = 0$

(i) $\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$ 3

$= 0 + \int_0^1 1\, dy + \int_1^0 (-dx) + 0 = 2$ ✓

Setter: R.C.J

Setter's signature:

Checker: J.D. G.

Checker's signature: J.D. Gibbon

E10

PAPER

QUESTION

SOLUTION C5

Please write on this side only, legibly and neatly, between the margins

(ii) $\oint_C (P\,dx + Q\,dy) = \iint_R (2x + 2y)\,dx\,dy$ $\quad P = -y^2$, $Q = x^2$

$= 2\int_0^1 \left[\int_0^1 (x+y)\,dx\right] dy$

$= 2\int_0^1 \left(\frac{x^2}{2} + xy\right)_0^1 dy = 2\int_0^1 \left(\frac{1}{2} + y\right) dy$

$= 2\left(\frac{1}{2}y + \frac{y^2}{2}\right)_0^1 = 2\cdot\left(\frac{1}{2} + \frac{1}{2}\right) = 2$ 4

15

Setter : RLJ

Checker : JDG

Setter's signature :

Checker's signature : J.D. Gibbon

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2004-2005

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PAPER II(3) E

QUESTION

SOLUTION 11

i.

$P(\textit{system down}) = P(\textit{A down}) + P(\textit{A not down} \cap \textit{B down})$

$= \frac{1}{3} + (\frac{2}{3} \times \frac{1}{3}) = \frac{5}{9} = 0.5556.$ 2

$P(\textit{system active}) = \frac{1}{6} \times \frac{1}{3} = \frac{1}{18} = 0.0556$ 2

$P(\textit{system dormant}) = 1 - \frac{5}{9} - \frac{1}{18} = \frac{7}{18} = 0.3889.$ 2

$E(\textit{cost}) = (2+3)P(\textit{A down} \cap \textit{B down}) + (1+3)P(\textit{A dormant} \cap \textit{B down})$

$+(2+1)P(\textit{A down} \cap \textit{B dormant})$

$+(1+1)P(\textit{A dormant} \cap \textit{B dormant})$ 4

$= 5(\frac{1}{3} \times \frac{1}{3}) + 4(\frac{1}{2} \times \frac{1}{3}) + 3(\frac{1}{3} \times \frac{1}{3}) + 2(\frac{1}{2} \times \frac{1}{3}) = \frac{17}{9}$ (=1.8889) 2

$E(\textit{cost} \mid \textit{B dormant}) = 1 + 2P(\textit{A down}) + P(\textit{A dormant})$

$= 1 + \frac{2}{3} + \frac{1}{2} = \frac{13}{6}$ (=2.1667) 3

Setter: M.J.Crowder — Setter's signature: M J Crowder

Checker: R. Coleman — Checker's signature: R Coleman

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2004-2005

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PAPER II(3)E

QUESTION

SOLUTION 12

	$X_2 = -1$	0	1	
$X_1 = 0$	1/15	2/15	2/15	1/3
1	4/15	2/15	4/15	2/3
	5/15	4/15	6/15	1

4

ii) $P(X_2 \geq 0) = 10/15$ (=0.6667) ; 1

$P(X_2 < X_1) = 7/15$ (=0.4667) ; 1

$P(X_1 = 0 \mid X_2 \geq 0) = 4/10$. 2

iii) means: $E(X_1) = \sum x_1 p(x_1) = \frac{2}{3}$ (=0.6667); 1

$E(X_2) = \frac{6}{15} - \frac{5}{15} = \frac{1}{15}$ (=0.0667); 1

$ratio = \frac{1}{10}$. 1

variances: $\text{var}(X_1) = \sum x_1^2 p(x_1) - E(X_1)^2 = \frac{2}{3} - \frac{4}{9} = \frac{2}{9}$ (=0.222) 2

$\text{var}(X_2) = \frac{11}{15} - \frac{1}{225} = \frac{164}{225} = 0.7289$, $ratio = \frac{82}{25} = 3.28$, 2

Setter: M.J.Crowder — Setter's signature: MJ Crowder

Checker: R. Coleman — Checker's signature: R Coleman