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UNIVERSITY OF LONDON [I(2)E 2003]

B.ENG. AND M.ENG. EXAMINATIONS 2003

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)

Thursday 5th June 2003 10.00 am - 1.00 pm

Answer EIGHT questions.

Corrected Copy

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

[I(2)E 2003]

1. Find the stationary points of the function

$$f(x,\, y) \;=\; y^4 + 4x^2y^2 - 2y^2 + 2x^2 - 1 \ .$$

Show that $f(x,\, y)$ has two minima and one saddle point.

Sketch some typical contours $f(x,\, y) = \text{constant}.$

2. If $x = r\cosh\theta$ and $y = r\sinh\theta$ and f is an arbitrary function of $(x,\, y)$, show that

$$\frac{\partial f}{\partial x} \;=\; \cosh\theta\,\frac{\partial f}{\partial r} \;-\; \frac{\sinh\theta}{r}\,\frac{\partial f}{\partial \theta}$$

and find a similar expression for $\dfrac{\partial f}{\partial y}$.

Verify these relationships for the case when $f(x,\, y) = \sqrt{x^2 - y^2}\,\tanh^{-1}(y/x)$.

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3. Compute an approximation to the integral

$$I = \int_0^{\pi/4} \theta \sin\theta d\theta$$

using:

(i) the Trapezium Rule with 2 ordinates (i.e. one strip);

(ii) the Trapezium Rule with 3 ordinates (i.e. two strips).

From your two approximations, give an improved estimate of the integral using Richardson extrapolation.

Verify by direct calculation that your improved estimate is the same as that from Simpson's rule with 3 ordinates.

Compare your results with the exact value of the integral.

You should work to 4 decimal places throughout.

[Richardson extrapolation: Let $I = \int_a^b f(x)dx$ and let I_1 and I_2 be two estimates for I obtained using the Trapezium Rule with intervals h and $h/2$. Then, provided h is small enough, $(4I_2 - I_1)/3$ is a better estimate of I.]

4. It is required to solve the equation

$$\tan x = \sqrt{a^2 - x^2}$$

using the Newton-Raphson iteration method.
By sketching the two curves $y = \tan x$ and $x^2 + y^2 = a^2$ with $y > 0$, show that this equation has 3 roots if $\pi < a < \dfrac{3\pi}{2}$.

If $a = 4$, use the Newton-Raphson method to find both positive roots correct to 4 decimal places. You should take as a first estimate $x_0 = 1.3$ for the smaller root and $x_0 = 3.8$ for the larger root and work to 5 decimal places throughout.

5. Two planes are defined by the equations

$$\begin{aligned} x + y - 2z &= 3 \,, \\ 2x - 2y + z &= 1 \,. \end{aligned}$$

(i) Find the perpendicular distance from the origin to each plane.

(ii) Find the vector equation of the straight line defined by the intersection of these planes.

6. (i) Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ;$$

calculate A^2 and hence find all 2×2 matrices A such that $A^2 = I$ (where I is the 2×2 identity matrix). Consider separately the cases where

(a) $b = c = 0$;

(b) neither b nor c is 0 ;

(c) exactly one of b or c is 0 .

(ii) Let A and B be general 2×2 matrices. Decide whether or not each of the following rules is valid:

(a) $(A + B)(A - B) = A^2 - B^2$;

(b) $(AB)^{-1} = B^{-1}A^{-1}$ (assuming that all 3 inverses exist);

(c) $(A + B)^{-1} = B^{-1} + A^{-1}$ (assuming that all 3 inverses exist).

Justify your answers briefly.

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7. Complete the factorisation

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & 2 \\ 1 & a & 2a \end{pmatrix} = LU = \begin{pmatrix} 1 & 0 & 0 \\ . & 1 & 0 \\ . & . & 1 \end{pmatrix} \begin{pmatrix} . & . & . \\ 0 & . & . \\ 0 & 0 & . \end{pmatrix}$$

where each dot denotes an element to be found.

Use the factorisation to solve the system of equations

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$$

for x_1, x_2, x_3 in terms of a and b (for $a \neq 1$).

Find also all solutions (if they exist) of the system in the cases:

(i) $a = 1, \;\; b \neq 0$;

(ii) $a = 1, \;\; b = 0$.

8. (i) Find the solution of the differential equation for $y(x)$

$$(y^2 - x^2) \frac{dy}{dx} = x^2 + 2xy ,$$

for which $y = 2$ when $x = 1$.

(ii) Find the general solution of the differential equation

$$(x+1) \frac{dy}{dx} - 3y = (x+1)^5 .$$

9. (i) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x .$$

(ii) Find the solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = \cos x ,$$

subject to the initial conditions $y(0) = 1,\ y'(0) = 0$.

10. Prove that for $0 \leq x < \pi$:

(i) $$x = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2}\dots\right)$$

and

(ii) $$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4}\dots\right) .$$

Use these equations to show that

(iii) $$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \pi^2/8 .$$

END OF PAPER

M A T H E M A T I C S   D E P A R T M E N T

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary}, |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\cos b + \cos a\sin b\,;$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!,\quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$