Master

E1.10 Mathematics I

# UNIVERSITY OF LONDON

## B.ENG. AND M.ENG. EXAMINATIONS 2006

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

**PART I : MATHEMATICS 1 (ELECTRICAL ENGINEERING)**

**Tuesday 30th May 2006 10.00 am - 1.00 pm**

*Answer EIGHT questions.*

Corrected Copy

*[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]*

1. (i) Consider the Heaviside function

$$H(x) = \begin{cases} 1; & x \geq 0, \\ 0; & x < 0. \end{cases}$$

(a) Where is $H$ discontinuous?

(b) Sketch the even and odd parts of $H(x) - H(x-1)$.

(ii) Consider the function

$$f(x) = x + 1/x.$$

Give a reasonable domain of definition of $f$ and the corresponding range.

Is this function even, odd or neither?

Show that $f(x) \geq 2$ if $x > 0$ and give a domain such that $f$ can be restricted to be an invertible function on that domain.

2. (i) The implicit relationship

$$x^2 - y^2 = 1$$

holds on some curve $\Gamma$ in the $xy$-plane.

Sketch $\Gamma$, noting any asymptotes and extreme values taken by $x$ and $y$ on $\Gamma$.

(ii) Sketch the graph of

$$y(x) = 2 + \frac{1}{1-x},$$

noting any important features.

PLEASE TURN OVER

3. (i) Differentiate with respect to $x$

$$\ln\left[x + (1+x^2)^{1/2}\right]\,; \qquad (\sin x)^x\,.$$

(ii) Given that

$$x(t) = t + \sin t \quad \text{and} \quad y(t) = t + \cos t\,,$$

find $\dfrac{dy}{dx}$ in terms of $t$ and show that

$$(1 + \cos t)^3\,\frac{d^2y}{dx^2} = \sin t - \cos t - 1\,.$$

4. (i) Given the function $f(x) = x^2 \sin x$, express the $n$-th derivative of $f$ in the form

$$\frac{d^n f}{dx^n} = A\cos x + B\sin x\,,$$

where $A$ and $B$ are coefficients that depend on $x$ and $n$.

(ii) Use induction to prove that

$$\sum_{n=0}^{N} x^n = \frac{x^{N+1}-1}{x-1}$$

for all integer $N$ and real $x \neq 1$.

5. Evaluate the following limits:

(i) $$\lim_{x \to \pi/4} \frac{\cos 2x}{\tan(\sqrt{x}) - 1} \ ;$$

(ii) $$\lim_{x \to \pi/4} \frac{\cos 2x}{\tan x - 1} \ ;$$

(iii) $$\lim_{x \to 2} \frac{\sqrt{(x+2)} - 2}{\sqrt{(x^3 - 4)} - 2} \ ;$$

(iv) $$\lim_{x \to \infty} \left( \frac{x+3}{x} \right)^x \ .$$

6. Evaluate the following integrals :

(i) $$\int \frac{1}{x^2 + x - 6} \, dx \ ;$$

(ii) $$\int_0^{\pi/2} (\sin^3 x \ - \ 3) \ \cos x \, dx \ ;$$

(iii) $$\int (1 \ - \ x)^{10} \, dx \ ;$$

(iv) $$\int \frac{1}{1 \ + \ \cos x \ + \ \sin x} \, dx \ .$$

**PLEASE TURN OVER**

7. Evaluate the following indefinite integrals :

(i) $$\int \frac{6x - 12}{\sqrt{x^2 - 4x + 5}}\, dx \;;$$

(ii) $$\int \frac{\sec^2 x}{4\tan x + 7}\, dx \;;$$

(iii) $$\int x^2 \sin x \, dx \;;$$

(iv) $$\int \frac{2x + 1}{x^2 - 5x + 6}\, dx \,.$$

8. (i) Show that the power series

$$\sum_{n=1}^{\infty} n!\, x^n$$

converges only if $x = 0$.

(ii) Find the radius of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{\ln(n)}{n+1}\, x^n \,.$$

(iii) Use the integral test to decide whether or not

$$\sum_{n=5}^{\infty} \frac{1}{\sqrt{n}}$$

converges.

9. (i) Express each of the following complex numbers in the form $a + ib$ with $a$ and $b$ real :

$$\text{(a)}\quad i^{105}\ ; \qquad \text{(b)}\quad \frac{1}{i}\ ; \qquad \text{(c)}\quad (1 - i\sqrt{3})^2\ .$$

(ii) Find all values of $z$ such that

$$\text{(a)}\quad e^z = 1\ ; \qquad \text{(b)}\quad e^z = 1 + i\ .$$

(iii) Express the following complex numbers in polar coordinates :

$$\text{(a)}\quad 1 + i\ ; \qquad \text{(b)}\quad -3 - i\ .$$

10. (i) Show directly from the definitions

$$\cosh(x) = \frac{1}{2}\,(e^x + e^{-x}) \quad \text{and} \quad \sinh(x) = \frac{1}{2}\,(e^x - e^{-x})$$

that

$$\cosh(nx) = \frac{1}{2}\,([\cosh x + \sinh x]^n + [\cosh x - \sinh x]^n)$$

for any integer $n \geq 1$ and verify the result independently for $n = 2$ .

(ii) Prove that

$$\sinh^{-1}(x) = \ln[x + \sqrt{x^2 + 1}]\ .$$

(iii) Sketch the functions

(a) $y = \sinh^{-1}(x)$ , (b) $y = \cosh^{-1}(x)$ and (c) $y = \tanh^{-1}(x)$ ,

where

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}\ .$$

**END OF PAPER**

MATHEMATICS DEPARTMENT

# MATHEMATICAL FORMULAE

## 1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\, \mathbf{b},\, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

## 2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$



## 3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i\sinh z\,; \quad \sinh iz = i\sin z\,.$$

## 4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\,D^{n-1}g + \ldots + \tbinom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x,\, y)$ about $(a,\, b)$:

$$f(a+h,\, b+k) = f(a,\, b) + [hf_x + kf_y]_{a.b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a.b} + \ldots$$

(d) Partial differentiation of $f(x,\, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u,\, v)$, $y = y(u,\, v)$, then $f = F(u,\, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x,\, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let $(a,\, b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D > 0$ and $f_{xx}(a,\, b) < 0$, then $(a,\, b)$ is a maximum;
If $D > 0$ and $f_{xx}(a,\, b) > 0$, then $(a,\, b)$ is a minimum;
If $D < 0$ then $(a,\, b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $\quad I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\, y)dx + Q(x,\, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y\,.$

## 5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$ :
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

## 6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let $I_1$, $I_2$ be two estimates of $I$ obtained by using Simpson's rule with intervals $h$ and $h/2$.

Then, provided $h$ is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of $I$.



## 7. LAPLACE TRANSFORMS

| Function | Transform | Function | Transform |
|---|---|---|---|
| $f(t)$ | $F(s) = \int_0^\infty e^{-st} f(t)dt$ | $af(t) + bg(t)$ | $aF(s) + bG(s)$ |
| $df/dt$ | $sF(s) - f(0)$ | $d^2f/dt^2$ | $s^2F(s) - sf(0) - f'(0)$ |
| $e^{at}f(t)$ | $F(s-a)$ | $tf(t)$ | $-dF(s)/ds$ |
| $(\partial/\partial\alpha)f(t,\alpha)$ | $(\partial/\partial\alpha)F(s,\alpha)$ | $\int_0^t f(t)dt$ | $F(s)/s$ |
| $\int_0^t f(u)g(t-u)du$ | $F(s)G(s)$ | | |
| $1$ | $1/s$ | $t^n (n = 1, 2 \ldots)$ | $n!/s^{n+1}$, $(s > 0)$ |
| $e^{at}$ | $1/(s-a)$, $(s > a)$ | $\sin\omega t$ | $\omega/(s^2+\omega^2)$, $(s > 0)$ |
| $\cos\omega t$ | $s/(s^2+\omega^2)$, $(s > 0)$ | $H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$ | $e^{-sT}/s$, $(s, T > 0)$ |

## 8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

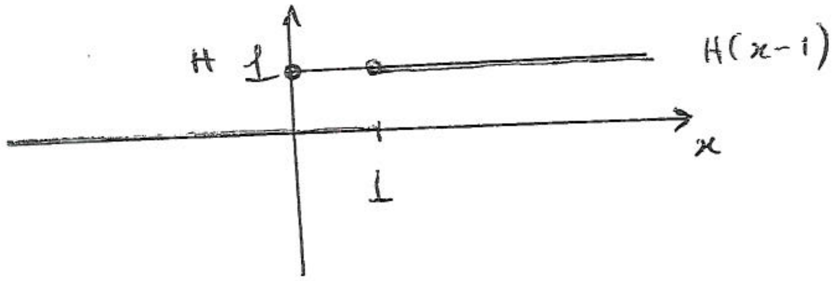
Parseval's theorem

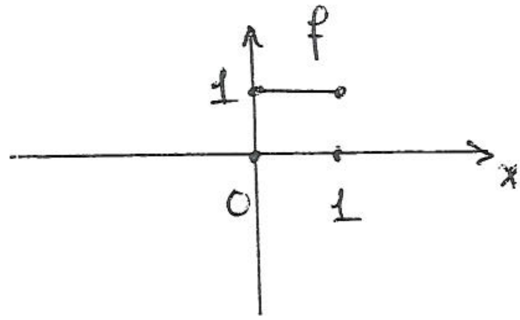
$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

1). a)

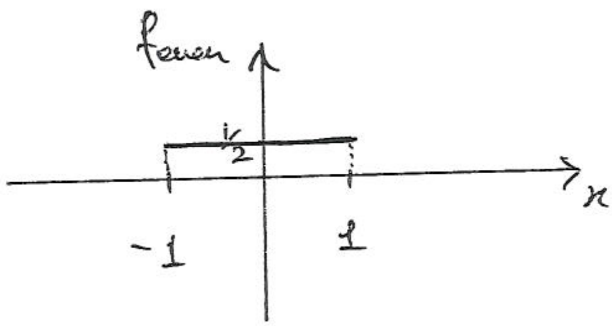
(i) The Heaviside function is discontinuous at $x = 0$ $\boxed{2}$

(ii) The even part of $f(x) \equiv H(x) \pm H(-1+x)$ can be found thus:

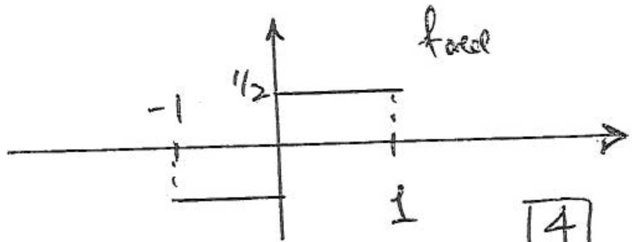



Then $f_{even}(x) = \dfrac{f(x) + f(-x)}{2}$




and $f_{odd}(x) = \dfrac{f(x) - f(-x)}{2}$




$\boxed{4}$

ii) If $f(x) = x + 1/x$ then $\{x \in \mathbb{R} \mid x \neq 0\}$ is one possible choice for $\mathcal{D}(f)$. $\boxed{2}$ Then

$f(-x) = -x + 1/_{-x} = -(x + 1/x) = -f(x)$ so $f$ is odd. $\boxed{2}$

Now $f(x) \sim x \to \infty$ as $x \to \infty$ and $f(x) \to +\infty$ as $x \to 0_+$

and $f'(x) = 1 - 1/x^2$ which is zero if $x = \pm 1$. $\boxed{4}$

At $x = +1$, $f(x) = 2$ and so $\min_{x>0} f(x) = 2 = f(1)$. $\boxed{2}$

Hence the range of $f$ with the above domain is $(-\infty, -2) \cup (2, \infty)$

If we define $D_{inv}(f) = \{x \in \mathbb{R} \mid x > 1\}$ then

$f'(x) > 0$ for all $x \in D_{inv}(f)$ and so $f$ is monotonic increasing on this domain and hence invertible. $\boxed{4}$ RB

2). a) Given $x^2 - y^2 = 1$, we can consider $y$ as a function of $x$ or $x$ as a function of $y$ on $\Gamma$. Now 2

$$x - y \cdot \frac{dy}{dx} = 0$$

and so $\frac{dy}{dx} = 0$ at $x = 0$ and so $y^2 = -1$!

Also,

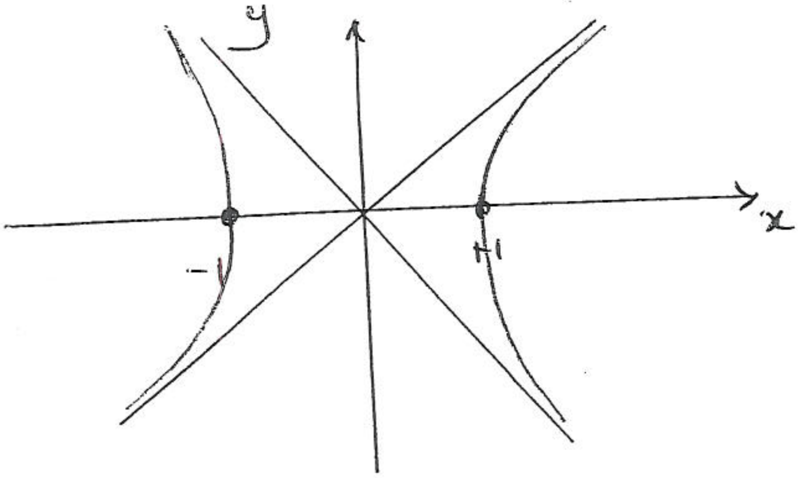
$$x \frac{dx}{dy} - y = 0 \quad \text{so} \quad \frac{dx}{dy} = 0 \text{ at } y = 0.$$

so $x = \pm 1$. $\boxed{2}$

$\boxed{2}$

Along $\Gamma$, $y = \pm\sqrt{x^2 - 1} = \pm |x| \sqrt{1 - x^{-2}}$ $(x^2 - 1 \geq 0)$

and so $y$ asymptotes to $\pm |x|$ as $x \to \infty$. $\boxed{4}$




$\boxed{4}$

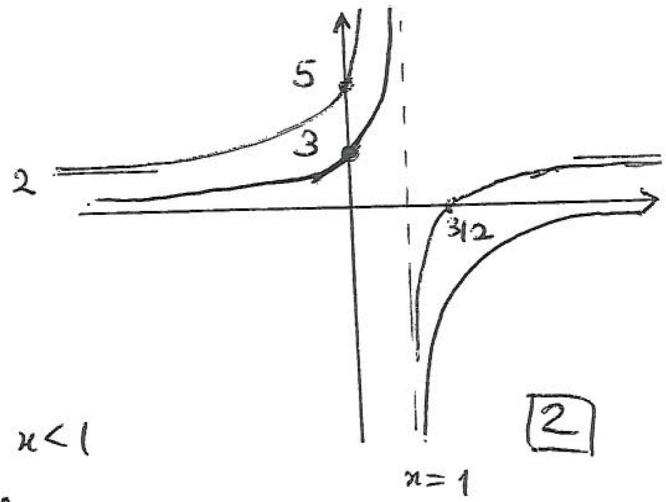
RB

2b). If $y(x) = 2 + \frac{1}{1-x} = \frac{2-2x+1}{1-x} = \frac{3-2x}{1-x}$.

Hence there is a vertical asymptote at $x=1$. $\boxed{2}$

Then $\frac{dy}{dx} = (1-x)^{-2}$ which is never zero $\boxed{2}$

and $\frac{1}{1-x}$ has the graph




$\boxed{2}$

and $\frac{d^2y}{dx^2} = 2((1-x)^{-3}$ which

means the graph is convex for $x<1$ and concave for $x>1$. $\boxed{1}$ (one) Finally,

$y(x)=0$ if $2+\frac{1}{1-x}=0$ so $x=3/2$. $\boxed{1}$

RB

# EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course: EE I (1) 3

Question: C1

Marks & seen/unseen

Parts

(i) $y = \ln\left[x + (1+x^2)^{1/2}\right]$

$$\frac{dy}{dx} = \frac{1 + x(1+x^2)^{-1/2}}{x + (1+x^2)^{1/2}} \qquad 3$$

$$\text{so } \frac{dy}{dx} = (1+x^2)^{-1/2}\left(\frac{(1+x^2)^{1/2} + x}{x + (1+x^2)^{1/2}}\right) = (1+x^2)^{-1/2} \qquad 2$$

$$y = (\sin x)^x \qquad \therefore \ln y = x \ln(\sin x) \qquad 2$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \ln(\sin x) + x\cot x$$

$$\text{so } \frac{dy}{dx} = (\sin x)^x\left[\ln(\sin x) + x\cot x\right] \qquad 2,1$$

(ii) $x = t + \sin t, \quad y = t + \cos t$

$$\text{so } \frac{dx}{dt} = 1 + \cos t, \quad \frac{dy}{dt} = 1 - \sin t \qquad 1,1$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-\sin t}{1+\cos t} \qquad 2$$

$$\text{So } (1+\cos t)\frac{dy}{dx} = 1 - \sin t \qquad 1$$

$$-\sin t \frac{dy}{dx} + (1+\cos t)\frac{d^2y}{dx^2}\frac{dx}{dt} = -\cos t \qquad 2$$

$$(1+\cos t)^2\frac{d^2y}{dx^2} = \sin t\left(\frac{1-\sin t}{1+\cos t}\right) - \cos t$$

$$= \frac{(\sin t - \sin^2 t - \cos t - \cos^2 t)}{1+\cos t}$$

$$\therefore (1+\cos t)^3\frac{d^2y}{dx^2} = \sin t - \cos t - 1 \qquad 3$$

Setter's initials: JRC

Checker's initials: SL

Page number

4a) If $f(x) = x^2 \sin x$ then

$$D^n f = \sum_{j=0}^{n} D^n(x^2 \sin x) = \sum_{j=0}^{n} {}^nC_j \, D^j(x^2) D^{n-j} \sin x$$

$$= \sum_{j=0}^{2} {}^nC_j \, D^j(x^2) \, D^{n-j}(\sin x)$$

$$= {}^nC_0 \, x^2 D^n(\sin x) + {}^nC_1 \, 2x \, D^{n-1}(\sin x)$$

$$+ {}^nC_2 \cdot 2 \cdot D^{n-2}(\sin x)$$

$$= x^2 D^n(\sin x) + 2nx \, D^{n-1}(\sin x) + n(n-1) D^{n-2}(\sin x)$$

$\boxed{6}$

If $n$ is even, this equals

$$x^2 D^n \sin x + 2nx \, D^{n-1}(-D \cos x) + n(n-1) D^{n-2}(-D^2 \sin$$

$$= x^2 D^n \sin x - 2nx \, D^n \cos x - n(n-1) D^n \sin x$$

$$= (x^2 - n(n-1)) D^n \sin x - 2nx \, D^n \cos x$$

$$= \begin{cases} (x^2 - (n-1)n) \cdot (-1)^{n/2} \sin x - 2nx \cdot (-1)^{n/2} \cos x \, ; \ n \text{ even} \\ (x^2 - n(n-1)) \cdot (-1)^{(n-1)/2} \cos x - 2nx \cdot (-1)^{(n+1)/2} \sin x \, ; \ n \text{ odd.} \end{cases}$$

$\boxed{4}$

RB.

(4)

b). To prove $\sum_{n=0}^{N} x^n = \frac{x^{N+1}-1}{x-1}$ for $x \neq 1$,

let $P(N)$ denote this as a proposition. Then

(i) $P(1)$ is true because

$$\sum_{n=0}^{1} x^n = 1 + x = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{(x-1)}$$

$$= 1 + x.$$ [4]

(ii) Assume $P(k)$ to be true. Then

$$\sum_{n=0}^{k+1} x^n = \sum_{n=0}^{k} x^n + x^{k+1} = \frac{x^{k+1}-1}{x-1} + x^{k+1}$$

$$= \frac{x^{k+1}-1 + x^{k+1}(x-1)}{x-1}$$

$$= \frac{x^{k+2}-1}{x-1}.$$

Hence $P(k+1)$ follows and so $P(k)$ is true for all $k \in \mathbb{N}$ by induction. [6]

R.B

| | EXAMINATION QUESTIONS/SOLUTIONS 2005-06 | Course EEI(1) 5 |
|---|---|---|
| Question 2 Soln | | Marks & seen/unseen |
| Parts | (i) When $x \to \pi/4$ numerator is zero, denominator is not. $\therefore$ limit $= 0$ | 3 |
| | (ii) Denominator and numerator both zero. $\therefore$ L'Hopital. $$\lim = \lim_{x\to\pi/4} \frac{-2\sin 2x}{\sec^2 x} = \frac{-2}{(2/\sqrt{2})^2} = -1$$ | 5 |
| | (iii) Again L'Hopital's rule $$\lim = \lim_{x\to 2} \frac{1/2(x+2)^{-1/2}}{1/2(x^3-4)^{-1/2}.3x^2}$$ $$= 1/12$$ | 5 |
| | Let $y = \left(\frac{x+3}{x}\right)^x$. Take logs $$\therefore \ln y = x\ln\left(\frac{x+3}{x}\right) = x\ln(1+3/x)$$ For large $x$ $\ln y = x\left(3/x - 9/2x^2 \cdots\right) = 3 - 9/2x \cdots$ So $\lim_{x\to\infty} \ln y = 3$ Hence $\lim_{x\to\infty} y = e^3$ | 7 |
| | Setter's initials gnc. | Checker's initials | Page number |

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|---|---|---|
| ~~Question~~ C3 | SOLUTION | Marks & seen/unseen |
| Parts | i) Using partial fractions<br>$\frac{1}{x^2+x-6} = \frac{1}{(x+3)(x-2)} = \frac{-1/5}{x+3} + \frac{1/5}{x-2}$<br>so<br>$\int \frac{1}{x^2+x-6} = -\int \frac{1/5}{x+3} + \int \frac{1/5}{x-2} =$<br>$= -\frac{1}{5}\ln\lvert x+3\rvert + \frac{1}{5}\ln\lvert x-2\rvert + C$<br>$= \frac{1}{5}\ln\left\lvert\frac{x-2}{x+3}\right\rvert + C$ | 5 |
| | ii) Let $t = \sin x$ and $dt = \cos x\, dx$.<br>Then<br>$\int_0^{\pi/2} (\sin^3 x - 3)\cos x\, dx = \int_0^1 t^3 - 3\, dt$<br>$= \left[\frac{t^4}{4} - 3t\right]_0^1 = \left(\frac{1}{4} - 3\right) - 0 = -2.75$ | 5 |
| | Setter's initials SC | Checker's initials | Page number 1/2 |

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|---|---|---|
| Question C3 | | Marks & seen/unseen |

Parts

iii) Let $u = 1-x$. Then $\frac{du}{dx} = -1$ and $dx = -du$. Therefore

$$\int (1-x)^{10} dx = \int -u^{10} du = -\frac{u^{11}}{11}$$

$$= -\frac{(1-x)^{11}}{11}$$

5

iv) Using $t = \tan(x/2)$ we have

$$\sin x = \frac{2t}{1+t^2} \quad \text{and} \quad \cos x = \frac{1-t^2}{1+t^2}$$

and so $\frac{dx}{dt} = \frac{2}{1+t^2}$ so

$$\int \frac{1}{1+\cos x + \sin x} dx = \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt$$

$$= \int \frac{2}{1+t^2+2t+1-t^2} dt = \int \frac{2}{2+2t} dt$$

$$= \int \frac{1}{1+t} dt = \ln(1+t) = \ln\left(1+\tan\frac{x}{2}\right)$$

5

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(i) Recognise that $\frac{d}{dx}(x^2-4x+5) = 2x-4$.

So that substitution $u = x^2-4x+5$

$\Rightarrow$ integral is $\int \frac{3\,du}{\sqrt{u}}$ — 2

$= 6\sqrt{u}$ + constant — 1

$\equiv 6\sqrt{x^2-4x+5}$ + constant — 2

(ii) Trigonometric substitution $v = \tan x$ where $dv/dx = \sec^2 x$ — 2

$\Rightarrow$ integral is $\int \frac{dv}{4v+7}$

$= \frac{1}{4}\ln|4v+7|$ + constant — 1

$\equiv \frac{1}{4}\ln|4\tan x+7|$ + constant — 2

(iii) Integrate by parts

$$\int x^2 \sin x\,dx = -x^2\cos x + \int 2x\cos x\,dx$$

$$= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + \text{constant}$$ 4

(iv) Note that $x^2 - 5x + 6 \equiv (x-2)(x-3)$
and use partial fractions

$$\frac{2x+1}{x^2-5x+6} \equiv \frac{2x+1}{(x-2)(x-3)} \equiv \frac{A}{x-2} + \frac{B}{x-3}.$$

$\Rightarrow 2x+1 \equiv A(x-3) + B(x-2)$

choose values or equate coeff $\Rightarrow$ $A+B=2$, $-3A-2B=1$ 3

$\Rightarrow A = -5,\ B = +7.$

$$\therefore \int \frac{(2x+1)\,dx}{x^2-5x+6} = \int \left(\frac{-5}{x-2} + \frac{7}{x-3}\right)dx$$

$$\equiv -5\ln|x-2| + 7\ln|x-3| + \text{constant}$$ 3

$$\left[\equiv \ln\left|\frac{(x-3)^7}{(x-2)^5}\right| + \text{constant}\right].$$

20

8a) Given the n-th term $a_n \equiv n!\, x^n$ Consider the limit-ratio $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right|$ (8)

and so define

$$\ell_x := \lim_{n\to\infty} |x|\, |n+1| = \infty \quad \text{for each } x \in \mathbb{R} \setminus \{0\}.$$

By the Limit Ratio test, the series only converges if $\ell_x \leq 1$. The result follows. $\boxed{6}$

b) Given $a_n = \frac{\ln(n)\, x^n}{n+1}$ then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |x| \cdot \left|\frac{\ln(n+1)}{\ln n} \cdot \frac{n}{n+1}\right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{\ln(n+1)}{\ln n}\right| \quad \text{as} \quad \lim_{n\to\infty} \frac{n}{n+1} = 1$$ $\boxed{4}$

and setting $z_n = \ln n$, we find

$$\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = \lim_{z\to\infty} \frac{\ln(e^z + 1)}{z} = \lim_{z\to\infty} \frac{\ln e^z(1+e^{-z})}{z}$$

$$= \lim_{z\to\infty} 1 + \frac{\ln(1+e^{-z})}{z} = 1$$ $\boxed{4}$

and so if $|x| < 1$ the series converges.

c). Because $\int_a^\infty \frac{1}{\sqrt{x}}\, dx = \lim_{T\to\infty} 2x^{1/2}\Big]_a^T$

$$= \infty$$

for any $a > 0$, the series diverges $\boxed{6}$

R.B

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|---|---|---|
| ~~Question~~ Solution C4 | | Marks & seen/unseen |
| Parts | | |
| i) | a) $i^{105} = \cancel{1}\ i$ | $\cancel{1}$ 2 |
| | b) $\frac{1}{i} = \frac{i}{i^2} = -i$ | 3 |
| | c) $(1-i\sqrt{3})^2 = 1 + (i\sqrt{3})^2 - 2i\sqrt{3} = -2 - i\sqrt{3}\cdot 2$ | 3 |
| ii) | a) $e^z = 1$ for $z = i2\pi k$, $k \in \mathbb{Z}$ | 3 |
| | b) $e^z = 1+i$ for $z = \log\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi k\right)$, $k \in \mathbb{Z}$ | 3 |
| iii) | a) $1+i = \sqrt{2}\, e^{i\pi/4}$ | 3 |
| | b) $-3-i = \sqrt{10}\cdot e^{i\theta}$, $\theta = \pi + \tan^{-1} 1/3$ | 3 |
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4, (a) $\cosh x = \frac{1}{2}(e^{x} + e^{-x})$

$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$

$\cosh x + \sinh x = e^{x}$

$\cosh x - \sinh x = e^{-x}$

so $\cosh nx = \frac{1}{2}\left[e^{nx} + e^{-nx}\right]$

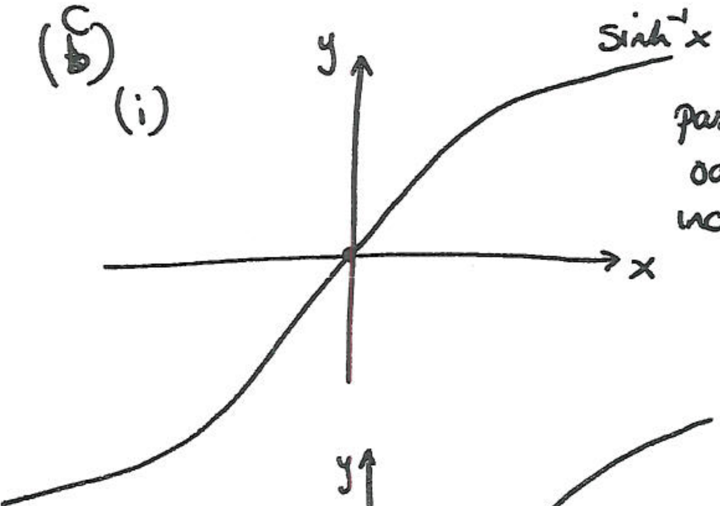
$= \frac{1}{2}\left[(\cosh x + \sinh x)^{n} + (\cosh x - \sinh x)^{n}\right]$ 4

if $n = 2$

$\cosh 2x = \frac{1}{2}\left[\cosh^{2}x + \sinh^{2}x + \cancel{2\cosh x \sinh x}\right.$
$\left. + \cosh^{2}x + \sinh^{2}x - \cancel{2\sinh x \cosh x}\right]$

$= \cosh^{2}x + \sinh^{2}x$ a standard result. 3

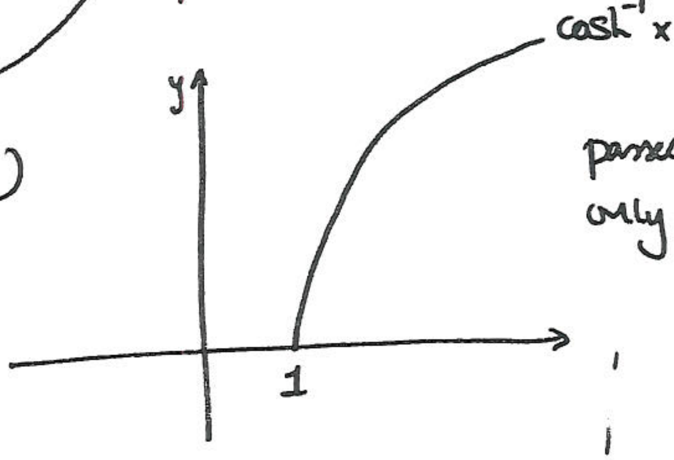
(c) (i)




passes through origin.
odd fn.
increases as $x \to \infty$

3

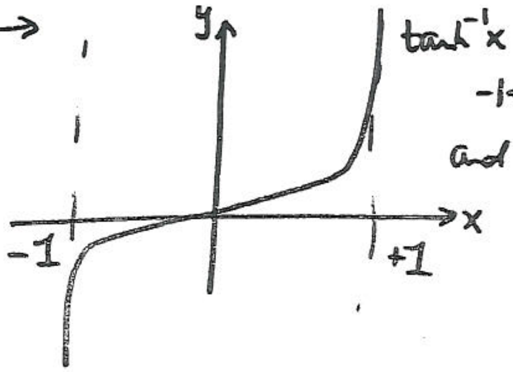
(ii)




passes thrugh 1.
only in $x > 1$

3

(iii)




$-1 < x < 1$
and $x \to \pm\infty$ as $x \to \pm 1$

3.

R.B

b)



$$y = \sinh x$$
$$= \tfrac{1}{2}(e^x + -e^{-x})$$

$$2y = e^x - e^{-x}$$

$$2ye^x = e^{2x} - 1$$

$$e^{2x} - 2ye^x + -1 = 0$$

$$(e^x)^2 - 2ye^x - 1 = 0$$

$$e^x = \frac{2y \pm \sqrt{4y^2+4}}{2}$$
$$= y \pm \sqrt{y^2+1}$$

Now $e^x > 0$ and $y - \sqrt{y^2+1} < 0$ so take $+$. 2 for this specific point.

$$e^x = y + \sqrt{y^2+1}$$

$\therefore\ x = \log(y + \sqrt{y^2+1})$. hence result. 2

R. B