UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 7 June, 2005 9 to 12

PAPER 53

THE STANDARD MODEL

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let g_i with $i = 1, 2, 3$ denote the coupling constants for the Standard Model gauge group factors $U(1)_Y$, $SU(2)$, $SU(3)$, respectively. The renormalisation group equation for each gauge coupling is, to one-loop order,

$$\mu \frac{dg_i}{d\mu} = \frac{\beta_i}{16\pi^2} g_i^3$$

where μ is the renormalisation scale and β_i is the one-loop beta-function coefficient (a real constant). Use this equation to relate $\alpha_i(M_Z)$ to $\alpha_i(\mu)$ where $\alpha_i \equiv g_i^2/4\pi$.

Grand Unified Theories predict that at some scale $\mu = M_{GUT}$,

$$\frac{5}{3}\alpha_1(M_{GUT}) = \alpha_2(M_{GUT}) = \alpha_3(M_{GUT}) \ .$$

Assuming this, derive

$$\frac{1}{\alpha_3(M_Z)} = \frac{1}{\alpha_2(M_Z)} + \frac{\beta_3 - \beta_2}{\frac{3}{5}\beta_1 - \beta_2} \left[\frac{3}{5\alpha_1(M_Z)} - \frac{1}{\alpha_2(M_Z)} \right] .$$

Write down a table giving the field content of one family of matter in the Standard Model together with one Higgs field. State clearly the spin and chirality (if appropriate) of each field and specify the representations of $U(1)_Y$, $SU(2)$ and $SU(3)$ under which the fields transform.

For an $SU(N)$ gauge theory with two-component (Weyl) fermions and complex scalars occurring in n_f and n_s copies of the fundamental (defining) representation respectively, the one-loop beta-function coefficient is

$$\beta_N = -\frac{11}{3}N + \frac{1}{3}n_f + \frac{1}{6}n_s$$

(there are no contributions from fields invariant under the gauge group). For $U(1)_Y$,

$$\beta_1 = \frac{2}{3}\sum_f Y_f^2 + \frac{1}{3}\sum_s Y_s^2$$

where Y_f and Y_s are the hypercharges of each two-component fermion and complex scalar. Use this information to calculate β_1, β_2 and β_3 in the Standard Model, indicating where each contribution comes from.

[*The electric charge is* $Q = T_3 + Y$, *where* T_3 *is the diagonal* $SU(2)$ *generator and* Y *the hypercharge.*]

2 The transformation rule for gauge fields is

$$T^a A^a_\mu \to U T^a A^a_\mu U^{-1} + \frac{i}{g}(\partial_\mu U) U^{-1}$$

where g is the gauge coupling. The covariant derivative is

$$D^i_{j\mu} = \partial_\mu \delta^i_j + i g A^a_\mu (T^a)^i_j$$

and the field strength is defined by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu \ .$$

What is U, and how can it be written explicitly in terms of representation matrices T^a?

In the case of QCD, how does a quark field ψ transform under the relevant gauge symmetry; what is the dimension of its representation; and what does that mean about the size of the matrices T^a? Write down the part of the QCD lagrangian which involves only the gluons, and hence derive the 3-gluon and 4-gluon interaction terms

$$\mathcal{L}_{glue} = -\frac{g^2}{4} f_{abc} f_{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} + g f_{abc} \partial_\mu A^a_\nu A^{b\mu} A^{c\nu} \ .$$

Now let g and g' denote the couplings corresponding to the electroweak gauge group $SU(2) \times U(1)_Y$. Write down the part of the Lagrangian which contains the electroweak covariant derivative acting on a left-handed quark doublet, expressing your answer in terms of the hypercharge gauge boson B_μ, the $SU(2)$ gauge bosons $\mathcal{A}^\alpha_\mu$, the Pauli matrices σ^α and quark fields u_L and d_L. The physical gauge bosons are defined by

$$W^\mu = (\mathcal{A}^\mu_1 - i\mathcal{A}^\mu_2)/\sqrt{2}, \qquad Z^\mu = \cos\theta_w \mathcal{A}^\mu_3 - \sin\theta_w B^\mu, \qquad \mathbf{A}^\mu = \cos\theta_w B^\mu + \sin\theta_w \mathcal{A}^\mu_3$$

where $\tan\theta_w = g'/g$. Show that the electromagnetic interaction terms for u_L and d_L are

$$\mathcal{L}_{em} = -\tfrac{2}{3} e \bar{u}_L \mathbf{A}^\mu \gamma_\mu u_L + \tfrac{1}{3} e \bar{d}_L \mathbf{A}^\mu \gamma_\mu d_L \ ,$$

where e is the electromagnetic coupling, which should be defined in terms of g and θ_w.

$$\left[\quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \right]$$

3 An approximate description of pion decay $\pi^-(p) \to e^-(k) + \bar{\nu}_e(q)$ is given by the Lagrangian

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}} J_\mu^\dagger J^\mu ,$$

where $J^\mu \equiv J_H^\mu + J_L^\mu$ has both hadronic and leptonic parts. The hadronic current is

$$J_H^\mu = \cos\theta_C \bar{u}\gamma^\mu(1-\gamma_5)d \equiv V_H^\mu - A_H^\mu ,$$

where V_H^μ and A_H^μ are vector and axial currents respectively and θ_C is the Cabibbo angle. The leptonic current is

$$J_L^\mu = \bar{\nu}_e\gamma^\mu(1-\gamma_5)e .$$

Explain briefly how the constant G_F emerges from more fundamental Standard Model parameters.

Show that the matrix element for pion decay may be written

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} \bar{u}_e(k)\gamma^\mu(1-\gamma_5)v_{\nu_e}(q)\langle 0|A_{H\mu}|\pi^-(p)\rangle \ .$$

Setting $\langle 0|\bar{u}\gamma^\mu\gamma_5 d|\pi^-(p)\rangle = i\sqrt{2}F_\pi p^\mu$, where F_π is a non-perturbative parameter (which can be measured), use the formula

$$\Gamma = \frac{1}{2m_\pi}\int \prod_i \Big(\frac{d^3q_i}{(2\pi)^3 2E_{q_i}}\Big)(2\pi)^4\delta^{(4)}\Big(p - \sum_i q_i\Big)\sum_s |\mathcal{M}|^2,$$

for decay into particles of momenta q_i and spins s, to derive the partial decay width

$$\Gamma(\pi^- \to e^-\bar{\nu}_e) = \frac{m_\pi}{4\pi} m_e^2 \left(1 - \frac{m_e^2}{m_\pi^2}\right)^2 G_F^2 F_\pi^2 \cos^2\theta_C .$$

What is the ratio of this expression to $\Gamma(\pi^- \to \mu^-\bar{\nu}_\mu)$? What does the agreement of experimental data with this ratio indicate?

[*You should assume the pion is odd under parity and that neutrinos are massless. You will find it helpful to work in the centre of mass frame and to use the gamma-matrix identities*

$$\gamma_5^2 = 1 \ , \qquad \mathrm{Tr}(\gamma\cdot k\,\gamma\cdot q) = 4k\cdot q \ , \qquad \mathrm{Tr}(\gamma_5\gamma\cdot k\,\gamma\cdot q) = \mathrm{Tr}(\gamma^\mu) = \mathrm{Tr}(\gamma_5\gamma^\mu) = 0 \ ,$$

as well as the result $\int dx\delta(f(x)) = \sum_{x_i} |f'(x_i)|^{-1}$ *where* $f(x_i) = 0$.]

4 The time-reversal operator $\hat{T}$ is anti-linear, obeying

$$\hat{T}(\alpha|\phi\rangle + \beta|\psi\rangle) = \alpha^*\hat{T}|\phi\rangle + \beta^*\hat{T}|\psi\rangle \ ,$$

and it also satisfies $\hat{T}^2 = 1$. By making appropriate assumptions about the action of $\hat{T}$ on momentum eigenstates, show that $\hat{T}$ is anti-unitary, obeying

$$\langle\hat{T}\phi|\hat{T}\psi\rangle = \langle\phi|\psi\rangle^* \ .$$

for any states $|\phi\rangle$ and $|\psi\rangle$.

Starting from the definition of the scattering matrix, or S-matrix,

$$S = \mathcal{T}\left(\exp\left[-i\int_{-\infty}^{\infty} dt V(t)\right]\right)$$

(where $\mathcal{T}$ denotes time-ordering), write down the condition on $V(t)$ for the theory to be time-reversal invariant, and show in detail that $\hat{T}S\hat{T}^{-1} = S^\dagger$. Explain how this leads to time-reversal invariant scattering amplitudes.

By considering the mode expansion of a complex scalar field, derive

$$\hat{T}a(p)\hat{T}^{-1} = \eta_T a(p_P),$$

where $a(p)$ annihilates a particle with 4-momentum $p^\mu = (p^0, \mathbf{p})$ and $p_P^\mu \equiv (p^0, -\mathbf{p})$. What is η_T and how can it be absorbed into the definition of the annihilation operator?

END OF PAPER