UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 8 June, 2006 1.30 to 4.30

PAPER 33

QUANTUM INFORMATION THEORY

Attempt ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Give the Bloch representation of the density matrix ρ of a qubit. What state does the origin of the Bloch sphere correspond to?

Suppose Alice prepares a qubit in either one of two pure states ρ_1 and ρ_2 where

$$\rho_i = |\psi_i\rangle\langle\psi_i|,$$

with

$$|\psi_1\rangle = |0\rangle \quad \text{and} \quad |\psi_2\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle.$$

She sends the qubit to Bob.

Find the Bloch vectors corresponding to these two states and determine the angle between them.

Suppose Bob does a measurement characterized by three POVM elements, E_1, E_2 and E_3, on the qubit that he receives from Alice. Given that

$$E_i = \frac{2}{3}|\phi_i\rangle\langle\phi_i| \quad \text{for} \quad i = 1, 2, 3,$$

where

$$|\phi_1\rangle = |1\rangle \text{ and } \quad |\phi_2\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle,$$

determine the POVM element E_3, and hence the vector $|\phi_3\rangle$.

Discuss the possible outcomes, their corresponding probabilities and Bob's conclusions. Justify that Bob never makes a mistake in identifying the state of the qubit.

2 A quantum operation is necessarily completely positive. Explain what is meant by complete positivity. Prove that a map Φ which is defined as follows:

$$\Phi(\rho) = \sum_k A_k \rho A_k^\dagger,$$

where ρ denotes a density matrix and the A_k are linear operators, is completely positive.

Prove that the map Φ defined by $\Phi(\rho) = \rho^T$, where T denotes transposition, is not completely positive.

3 Consider a 2-level atom A. If the atom is in its excited state then it has a probability p of decaying to its ground state. This decay is accompanied by the spontaneous emission of a photon.

(a) What is the quantum channel that can be used to model this atom? What are its Kraus operators? What process does each of these Kraus operators correspond to?

(b) The atom is originally in a state $\rho = \sum_{i,j=0}^{1} \rho_{ij} |i\rangle\langle j|$. Here $|i\rangle$, $i = 0, 1$, denote orthonormal basis states of the Hilbert space of the atom. By considering the unitary evolution of the atom and its environment, deduce how the state of the atom changes under the action of one use of the channel.

(c) What is the effect of two successive uses of the channel on the state ρ. Find the final state after n uses of the channel, in the limit $n \to \infty$. What is the von Neumann entropy of this state? Does it correspond to an information gain?

(d) What is a unital channel? Is this channel unital?

4 Quantum errors acting in an n-qubit Hilbert space $\mathcal{H}^{\otimes n}$ are characterized by an error vector $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_i \in \{I, X, Y, Z\}$. The corresponding operator is called a Pauli operator and is given by:

$$\mathbf{E}_{\underline{\alpha}} := \mathbf{E}_{\alpha_{j_1}}^{(j_1)} \cdots \mathbf{E}_{\alpha_{j_r}}^{(j_r)},$$

where the operators $\mathbf{E}_{\alpha_{j_i}}^{(j_i)}$ are defined by their actions on the basis vectors of $\mathcal{H}^{\otimes n}$. What are the actions of the operators $\mathbf{E}_X^{(j)}$, $\mathbf{E}_Z^{(j)}$ and $\mathbf{E}_Y^{(j)}$ on a basis state $|\underline{x}\rangle \in \mathcal{H}^{\otimes n}$, where $\underline{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$?

State the conditions under which a quantum error correcting code is **(a)** E error–correcting and **(b)** D error–detecting. What are the conditions under which the code corrects and detects these errors *non-degenerately*?

Using the above conditions, prove that an $[[n, k, d]]$ quantum error-correcting code detects $(d-1)$ errors and corrects $(d-1)/2$ errors. What is the maximum number of errors that the code could correct if their locations were known? Justify your answer.

The Shor code has the following basis codewords:

$$\begin{aligned} |\underline{0}\rangle &= \frac{1}{\sqrt{8}}\big(|000\rangle + |111\rangle\big)^{\otimes 3} \\ |\underline{1}\rangle &= \frac{1}{\sqrt{8}}\big(|000\rangle - |111\rangle\big)^{\otimes 3} \end{aligned} \qquad (1)$$

How would you use this code to correct a phase flip error? The Shor code corrects such an error *degenerately*. Explain why.

5 **(a)** State the strong subadditivity property of the von Neumann entropy. Use it to prove the following:

1. Discarding quantum systems never increases mutual information, i.e.,

$$S(\rho_A : \rho_B) \leq S(\rho_A : \rho_{BC}). \tag{2}$$

2. A quantum operation Φ can never increase mutual information i.e.,

$$S(\rho_A : \Phi(\rho_B)) \leq S(\rho_A : \rho_B). \tag{3}$$

(b) Suppose Alice has a classical source, characterized by a random variable X, which takes values $x \in J = \{1, 2, \ldots, N\}$ with probabilities $p(x)$. She encodes the symbol x into a quantum state ρ_x and sends it to Bob. Bob does a measurement on it, described by a finite set of POVM elements $\{E_y\}$. Let Y be the classical random variable corresponding to the outcome of the measurement.

(i) Describe how you can use three quantum systems to describe the above procedure. Let these systems be denoted as A, Q and B. What is the initial state ρ_{AQB} of the composite system AQB? Let A', Q' and B' denote the systems after Bob's measurement. What is the final state $\rho_{A'Q'B'}$?

(ii) Using the results proved in part **(a)**, prove that

$$S(A' : B') \leq S(A : Q). \tag{4}$$

(iii) Prove that (4) is the Holevo bound, clearly quoting any other result that you use.

END OF PAPER