UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 7 June 2001 9 to 12

PAPER 25

QUANTUM INFORMATION THEORY

*Attempt any **THREE** questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 For a pair of classical linear codes $\mathcal{C}$, $\mathcal{C}'$, with $\mathcal{C} \subset \mathcal{C}'$, describe the Calderbank–Shor–Steane construction giving a quantum code $\mathcal{X}$, and establish the correcting abilities of $\mathcal{X}$.

2 (The Bloch sphere) Prove that a density matrix ρ in the single-qubit Hilbert space $\mathcal{H}$ is a linear combination of the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence establish a 1–1 affine correspondence Φ between the density matrices in $\mathcal{H}$ and the points of the 3-dimensional unit ball $\mathbf{B}$ centered at the origin. [The affine property is that $\Phi(p_1\rho_1 + p_2\rho_2) = p_1\Phi(\rho_1) + p_2\Phi(\rho_2)$ for density matrices ρ_1, ρ_2 and non-negative numbers p_1, p_2 such that $p_1 + p_2 = 1$.] Prove that the pure state density matrices are mapped onto the unit sphere $\partial\mathbf{B}$. Finally, if ρ has eigenvectors ϕ_1 and ϕ_2 and eigenvalues λ_1 and λ_2, with $\lambda_1 \geqslant \lambda_2$, show that the pure state density matrices $|\phi_1\rangle\langle\phi_1|$ and $|\phi_2\rangle\langle\phi_2|$ correspond to vectors $\underline{n}_1, \underline{n}_2 \in \partial\mathbf{B}$, where $\underline{n}_1 = \frac{\underline{n}}{||\underline{n}||}$, $\underline{n}_2 = -\frac{\underline{n}}{||\underline{n}||}$ and $\underline{n} = \Phi(\rho)$.

3 Define the von Neumann entropy $S(\rho)$ of a density matrix ρ and prove its basic properties. Suppose ρ is a density matrix in $\mathcal{K}_1 \otimes \mathcal{K}_2$. Define the partial traces $\rho_1 = \mathrm{tr}_{\mathcal{K}_2}\rho$ and $\rho_2 = \mathrm{tr}_{\mathcal{K}_1}\rho$ and prove that ρ_1 and ρ_2 are density matrices in $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Prove that if ρ is a pure state density matrix then the von Neumann entropies $S(\rho_1)$ and $S(\rho_2)$ coincide.

4 Define the quantum Fourier transform using an expression analogous to the classical definition. Show that there exists an equivalent expression which can be efficiently computed using a small number of simple gates.